2. Representation theory and basic theorems

MATRIX REPRESENTATIVES

<u>Def</u>: A vectorial space on $\mathbb{R}$ (or $\mathbb{C}$) is a set V on which we have consistently defined 2 operation

- sum : $v_1, v_2 \in V \Rightarrow v_1 + v_2 = v_3$
- product with a scalar $a \in \mathbb{R}$ (or $\mathbb{C}$) $a v_1$

the operation must satisfy the following properties

V is an Abelian group with respect to $+$

$$\left\{\begin{array}{ll} \forall\, u\ v\ w \in V & (u+v)+w = u+(v+w) \\ \exists\, 0 \in V : \forall v \in V & v + 0 = 0 + v = v \\ \forall v \in V\ \exists -v \in V : & v + (-v) = (-v) + v = 0 \\ \forall\, v, w \in V & v + w = w + v \end{array}\right.$$

$$\forall \lambda \in \mathbb{R}\ (\text{or } \mathbb{C})\ \forall v, w \in V \quad \lambda(v+w) = \lambda v + \lambda w$$
$$\forall \lambda, \mu \in \mathbb{R}\ (\text{or } \mathbb{C})\ \forall v \in V \quad (\lambda+\mu) v = \lambda v + \mu v$$
$$\forall \lambda, \mu \in \mathbb{R}\ (\text{or } \mathbb{C})\ \forall v \in V \quad (\lambda\mu) v = \lambda(\mu v)$$
$$\forall v \in V \quad 1 v = v \quad \text{and} \quad 0 v = 0$$

Notice that $\mathbb{R}$ or $\mathbb{C}$ can be substituted with whatever other field $\mathbb{K}$.

<u>Def</u>: An application $f : V \to V$ is linear if:

$$f(\alpha v + \beta w) = \alpha f(v) + \beta f(w) \qquad \forall\, v, w \in V \quad \alpha, \beta \in \mathbb{K}.$$

In particular the following theorem holds. The space of linear transformations $V \to V$ is isomorphous to the space of matrices $n \times n$ with entries taken from $\mathbb{K}$.

$$\mathrm{Hom}_{\mathbb{K}}(V_n, W_m) \simeq \mathrm{Mat}(m, n; \mathbb{K})$$

Our point symmetry operations are all linear operations acting first of all on the points of $\mathbb{R}^3$ which is a vectorial space on $\mathbb{R}$.

$f: \mathbb{R}^3 \to \mathbb{R}^3$. In order to find the corresponding real 3×3 matrix is enough to take a basis for $\mathbb{R}^3$ and apply f to each of the element of the basis and decompose the result on the same basis.

$$\mathbb{R}^3 \to \quad \hat{e}_x \;\; \hat{e}_y \;\; \hat{e}_z \qquad \begin{aligned} f(\hat{e}_x) &= \hat{v}_x = a\hat{e}_x + b\hat{e}_y + c\hat{e}_z \\ f(\hat{e}_y) &= \hat{v}_y = d\hat{e}_x + e\hat{e}_y + f\hat{e}_z \\ f(\hat{e}_z) &= \hat{v}_z = g\hat{e}_x + h\hat{e}_y + i\hat{e}_z \end{aligned}$$

$$f(\hat{v}) = \hat{w} = \alpha'\hat{e}_x + \beta'\hat{e}_y + \gamma'\hat{e}_z$$

$$\parallel$$

$$f(\alpha\hat{e}_x + \beta\hat{e}_y + \gamma\hat{e}_z) = \alpha\hat{v}_x + \beta\hat{v}_y + \gamma\hat{v}_z = (\alpha a + \beta d + \gamma g)\,\hat{e}_x + (\alpha b + \beta e + \gamma h)\,\hat{e}_y + (\alpha c + \beta f + \gamma i)\,\hat{e}_z.$$

written in components: $$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

All the point symmetry operations that we consider preserve the scalar prod. defined on $\mathbb{R}^3$ $\qquad (v, w) = (f(v), f(w))$

This property transforms into a property of the matrix M associated to f. Namely M is unitary i.e. $MM^+ = M^+M = E$

proof:

We use the scalar product given by $(\hat{e}_i, \hat{e}_j) = \delta_{ij}$. The scalar product is a bilinear form => we have is automatically defined on the whole vectorial space.

$$(f(v), f(w)) = \left(f\left(\sum_i \alpha_i \hat{e}_i\right), f\left(\sum_j \beta_j \hat{e}_j\right)\right) =$$

$$= \sum_{ij} \alpha_i^* \beta_j \left(f(\hat{e}_i), f(\hat{e}_j)\right) =$$

$$= \sum_{ij} \alpha_i^* \beta_j \left(\sum_l M_{li} \hat{e}_l, \sum_k M_{kj} \hat{e}_k \right)$$

$$= \sum_{ij} \alpha_i^* \beta_j \sum_{lk} M_{li}^* M_{kj} \delta_{lk} = \sum_{ijk} \alpha_i^* \beta_j (M^+)_{ik} M_{kj}$$

$$(v, w) = \sum_{ij} (\alpha_i \hat{e}_i, \beta_j \hat{e}_j) = \sum_i \alpha_i^* \beta_i$$

$$\Rightarrow \sum_k (M^+)_{ik} M_{kj} = \delta_{ij}$$

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If the transformation is real => $M^T M = E$ and M is orthogonal.

Theo: Let $\{A, B, \dots\}$ form a group G => the set of matrix representation $\{\Gamma(A), \Gamma(B), \dots\}$ form a group isomorphous to G.

Proof: it is enough to prove the homomorphism.

$$\Gamma(AB) = \Gamma(A)\Gamma(B)$$

$$AB = C \quad \Rightarrow \quad C(\hat{v}) = \sum_{il} \alpha_i M_{li}^C \hat{e}_l$$

$$A(B(v)) = A\left(\sum_{il} \alpha_i M_{li}^B \hat{e}_l\right) = \sum_{il} \alpha_i M_{li}^B A(\hat{e}_l)$$

$$= \sum_{il} \alpha_i M_{li}^B M_{kl}^A \hat{e}_k$$

$$\sum_{ikl} \alpha_i M^A_{kl} M^B_{li}\, \hat{e}_k = \sum_{il} \alpha_i\, M^C_{li}\, \hat{e}_l \qquad \forall\, \{\alpha_i\}.$$

$$\sum_{lk} M^A_{lk} M^B_{ki}\, \hat{e}_l = \sum_{l} M^C_{li}\, \hat{e}_l$$ project on the 3 components

$$\sum_{k} M^A_{lk} M^B_{ki} = M^C_{li} \quad \Leftrightarrow \quad M^A M^B = M^C$$

<u>example</u> C_4 : $E,\ C_4^+,\ C_4^-,\ C_2$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad C_4^+ = \begin{pmatrix} 0 & \bar{1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad C_4^- = \begin{pmatrix} 0 & 1 & 0 \\ \bar{1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \bar{1} & 0 & 0 \\ 0 & \bar{1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \bar{1} \equiv -1.$$

These matrices act on the COEFFICIENTS!

TRANSFORMATION OF FUNCTIONS

Let's take a point P in the space identified by the coordinates xyz $R(P) = P'$ where P' is represented by the coordinates $x'y'z'$. If R is a symmetry operation then there must be a representation of R in the space of the functions f such that

$$\hat{R} f(R(P)) = f(P) \qquad P = R^{-1}(P')$$

$$\hat{R} f(P') = f(R^{-1}(P'))$$ but this is applicable to any point

$$\hat{R} f(P) = f(R^{-1}(P))$$ $\rightsquigarrow$ this represents a definition of the function transformation induced from the point symmetry operation R.

example

$$d_{xy} = e^{-q(r)}\, xy \qquad R = C_4^+$$

$$C_4^+ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \bar{y} \\ x \\ z \end{pmatrix} \qquad (C_4^+)^{-1} = C_4^- = (C_4^+)^T \Rightarrow (C_4^+)^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ \bar{x} \\ z \end{pmatrix}$$

$$\hat{C}_4^+ d_{xy} = e^{-q(r)}\, y\bar{x} = -d_{xy} .$$

N.B. • In quantum mechanics the relation $\hat{R} f(R(P)) = f(P)$ should be softened if f is a wave function since physical properties do not depend on the overall phase. $\hat{R}\psi(R(P)) = e^{i\varphi}\psi(P)$

Question : If $\{R, T, S, \ldots\}$ is a group, can we say the same for $\{\hat{R}, \hat{T}, \hat{S}, \ldots\}$. ?

We define the product as $\hat{S} = \hat{R}\hat{T}$ $\quad \hat{S} f = \hat{R}[\hat{T} f(P)]$. Once again it is crucial to prove that $\hat{S} \leftrightarrow S$ and all other relations follow immediately. On the top we prove in such a way that the 2 groups are isomorphic.

$$\hat{R}[\hat{T} f(P)] = \hat{R} f(T^{-1}(P)) = f(T^{-1}(R^{-1}(P))) = f((RT)^{-1}P) \qquad (*)$$

$$\hat{S}(f(P)) = f(S^{-1}(P)) = f((RT)^{-1}(P)) =$$

$\Rightarrow$ the relation is $RT \to \hat{R}\hat{T}$.

The operators $\hat{T}$ are linear acting on the Hilbert space for our system and for this reason they are also represented by matrices. The group of matrices that represent the transformations $\hat{T}$ is also isomorphous to the original point symmetry group. It is also possible to prove that every representation with matrices having non-vanishing determinant is similar to a representation made of unitary matrices.

<u>proof</u>

$A_1 \dots A_h$ are the matrices of a representation

$H = \sum_{x=1}^{h} A_x A_x^+$ is hermitian

$d = U^{-1} H U = \sum_x U^{-1} A_x A_x^+ U = \sum_x U^{-1} A_x U U^{-1} A_x^+ U = \sum_x \hat{A}_x \hat{A}^+$

and $A_x \approx \hat{A}_x$

$d_{kk} = \sum_x \sum_j (\hat{A}_x)_{kj} (\hat{A}_x^+)_{jk} = \sum_{xj} (\hat{A}_x)_{kj} (\hat{A}_x)^*_{kj} = \sum_{xj} |(\hat{A}_x)_{kj}|^2$

$d^{1/2}$ and $d^{-1/2}$ by taking the square root of the diagonal elements

$$d = d^{1/2} d^{1/2} = \sum_x \hat{A}_x \hat{A}_x^+$$

$$\hat{\hat{A}}_x \equiv d^{-1/2} \hat{A}_x d^{1/2} \qquad \hat{\hat{A}}_x^+ = d^{1/2+} \hat{A}_x^+ d^{-1/2+} = d^{1/2} \hat{A}_x^+ d^{-1/2}$$

$$\begin{aligned}
\hat{\hat{A}}_x \hat{\hat{A}}_x^+ &= d^{-1/2} \hat{A}_x d^{1/2} d^{1/2} \hat{A}_x^+ d^{-1/2} = d^{-1/2} \hat{A}_x \sum_y \hat{A}_y \hat{A}_y^+ \hat{A}_x^+ d^{-1/2} \\
&= d^{-1/2} U^{-1} \sum_y A_x A_y A_y^+ A_x^+ U d^{-1/2} = d^{-1/2} U^{-1} \sum_y (A_x A_y)(A_x A_y)^+ U d^{-1/2} \quad \Big] \text{ rearrangement theorem.} \\
&= d^{-1/2} U^{-1} \sum_z A_z A_z^+ U d^{-1/2} = d^{-1/2} \sum_z \hat{A}_z \hat{A}_z^+ d^{-1/2} = E
\end{aligned}$$

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<u>Theorem</u>: The function operators associated to the symmetry operations are unitary.

proof

$$\begin{aligned}
\hat{R} \hat{R}^+ \psi(P) &= \hat{R} \psi'(P) = \psi'(R^{-1}(P)) = \hat{R}^+ \psi(R^{-1}(P)) \\
&= \psi((R^+)^{-1}(R^{-1})(P)) = \psi((RR^+)^{-1}(P)) = \psi(E^{-1}(P)). \\
&= \hat{E} \psi(P).
\end{aligned}$$

This expression make sense only if $\hat{R}^+$ belongs to the group. Its action is by definition given starting from R^+!

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Another quantum mechanical consideration

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\downarrow$$

$$\hat{H}|\psi_i\rangle = E_i|\psi_i\rangle$$

$\hat{R}|\psi_i\rangle$ must be still an eigenstate o H with the same eigenvalue

$\langle x|\hat{R}|\psi_i\rangle = \hat{R}\,\psi_i(x) := \psi_i(R^{-1}(x))$ and the system is invariant under the transformation R (and thus R^{-1} in the coordinates)

$$\hat{H}\left(\hat{R}|\psi_i\rangle\right) = E_i\,\hat{R}|\psi_i\rangle = \hat{R}\,E_i|\psi_i\rangle = \hat{R}\hat{H}|\psi_i\rangle$$

$$\Rightarrow \qquad [\hat{H},\hat{R}] = 0$$

The set of operators that commute with $\hat{H}$ is called group of the Hamiltonian.

<u>Def</u> If by one and the same equivalence transformation, all the matrices in the representation of a group can be made to acquire the same block form, then the representation is said to be <u>reducible</u>. Otherwise it is <u>irreducible</u>. Thus an irreducible representation can not be expressed in terms of representations of lower dimensionality.

$$T \longrightarrow \begin{pmatrix} \Gamma_1(T) & 0 \\ 0 & \Gamma_2(T) \end{pmatrix}$$

$$T = RS \qquad \begin{pmatrix} \Gamma_1(R) & 0 \\ 0 & \Gamma_2(R) \end{pmatrix}\begin{pmatrix} \Gamma_1(S) & 0 \\ 0 & \Gamma_2(S) \end{pmatrix} = \begin{pmatrix} \Gamma_1(R)\Gamma_1(S) & 0 \\ 0 & \Gamma_2(R)\Gamma_2(S) \end{pmatrix}$$

There are a few fundamental theorems in group theory that are of great importance for its application:

<u>SCHUR'S LEMMA</u> (part 1) <u>1905</u>

A matrix which commutes with all matrices of an irreducible representation is a constant matrix. i.e. a constant times the unit matrix. Therefore, if a non-constant commuting matrix exists, the representation is reducible; if none exists, the representation is irreducible.

<u>proof</u>

$$M A_x = A_x M \qquad (*)$$

$$\Downarrow$$

$$A_x^+ M^+ = M^+ A_x^+$$

A_x can be taken without loosing generality as unitary. by multiplying from left by A_x and from the right with A_x

$$M^+ A_x = A_x M^+ \qquad (**)$$

(*) + (**) + bilinearity of $[\cdot,\cdot]$

$$\Rightarrow \quad [A_x, M + M^+] = 0 \qquad i[A_x, M - M^+] = 0 \qquad \left.\begin{array}{l} H_1 = M + M^+ \\ H_2 = i(M - M^+) \end{array}\right\} \text{Hermitian.}$$

If $[A_x, H_j] = 0$ Where H_j is a generic hermitian matrix $\Rightarrow$ $H_j = \lambda \mathbb{1}$.

$$d = U^{-1} H_j U \qquad \hat{A}_x = U^{-1} A_x U \quad \Rightarrow \quad [d, \hat{A}_x] = 0$$

$$d_{ii} (\hat{A}_x)_{ij} = (\hat{A}_x)_{ij} d_{jj} \quad \Rightarrow \quad (\hat{A}_x)_{ij} (d_{ii} - d_{jj}) = 0 \qquad \forall A_x$$

if $d_{ii} \neq d_{jj} \Rightarrow (\hat{A}_x)_{ij} = 0 \quad \forall x \quad \Rightarrow \quad \hat{A}_x$ are in the same block form.

But A_x and $\Rightarrow \hat{A}_x$ cannot be brought <u>all</u> together in the same block form since they are an irreducible representation.

$$\Rightarrow \quad d_{ii} = d_{jj} \quad \forall i,j. \quad \Rightarrow \quad M = \tfrac{1}{2}(H_1 - iH_2) = \lambda \mathbb{1}. \qquad \blacksquare$$

SCHUR'S LEMMA (part 2)

If the matrix representations $\Gamma^1(A_1), \Gamma^1(A_2), \dots, \Gamma^1(A_h)$ and $\Gamma^2(A_1), \dots, \Gamma^2(A_h)$ of a given group $\mathcal{G}$ are irreducible representation of dimensionality l_1 and l_2 respectively, then, if there is a matrix of l_1 columns and l_2 rows M such that

$$(*) \qquad M\Gamma^1(A_x) = \Gamma^2(A_x)M \qquad \forall A_x \in \mathcal{G}$$

$\Rightarrow$ $l_1 \neq l_2$ $\quad M = \mathbb{O}$ the null matrix

$l_1 = l_2$ $\quad M = \mathbb{O}$ or Γ^1 and Γ^2 are similar. (equivalent)

proof:

We can assume without loss of generality that $l_1 \leq l_2$. We start by taking the adjoint of $(*)$. Namely

$$[\Gamma^1(A_x)]^+ M^+ = M^+[\Gamma^2(A_x)]^+$$

We can always take the representations to be unitary $\Rightarrow [\Gamma(A_x)]^+ = \Gamma(A_x)^{-1} = \Gamma(A_x^{-1})$.

$$\Gamma^1(A_x^{-1})M^+ = M^+\Gamma^2(A_x^{-1}) \qquad (**)$$

Since $A_x^{-1} \in \mathcal{G}$ $\qquad M\Gamma^1(A_x^{-1}) = \Gamma^2(A_x^{-1})M \qquad (***)$

$$\Rightarrow \quad M\Gamma^1(A_x^{-1})M^+ \overset{(**)}{=} MM^+\Gamma^2(A_x^{-1}) \overset{(***)}{=} \Gamma^2(A_x^{-1})MM^+ \overset{SL1}{\Rightarrow} MM^+ = c_2\mathbb{1}$$

$$\Rightarrow \quad \Gamma^1(A_x^{-1})M^+M \overset{(**)}{=} M^+\Gamma^2(A_x^{-1})M \overset{(***)}{=} M^+M\,\Gamma^1(A_x^{-1}) \overset{SL1}{\Rightarrow} M^+M = c_1\mathbb{1}$$

- $l_1 = l_2$ $\quad$ 1- $c_1 \neq 0 \Rightarrow M$ is regular and $M^{-1} = \dfrac{M^+}{c_1} \Rightarrow c_1 = c_2$

  $\Gamma^1(A_x) = M^{-1}\Gamma^2(A_x)M \Rightarrow$ the 2 representations are equivalent.

  2- $c_1 = 0 \Rightarrow \forall i,j \leq l_1 \;\; 0 = \sum_k (M^+)_{ik}M_{kj} = \sum_k M_{kj}M^*_{ki}$

  $i=j \qquad 0 = \sum_k |M_{ki}|^2 \Rightarrow M = \mathbb{O}.$

  $\forall i$

- $l_1 < l_2$

$$N = \begin{array}{c} \\ l_2 \end{array}\left(\begin{array}{c|c} \overbrace{\quad}^{l_1\,col} & \overbrace{\quad}^{l_2 - l_1} \\ M & \begin{matrix} 0000 \\ 0000 \\ \vdots \\ 0000 \end{matrix} \end{array}\right)$$

$NN^+ = MM^+ = c_2\mathbb{1}$ (dimension $l_2 \times l_2$)

But $\det N = \det N^+ = 0 \Rightarrow 0 = c_2^{\,l_2}.\;\; c_2 = 0$

$\Rightarrow 0 = \sum_k |M_{ki}|^2 \Rightarrow \forall k\,i \;\; M_{ki} = 0.$

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There is an orthogonality theorem that is so central to the application of group theory to quantum mechanics that it was named the "Wonderful Orthogonality Theorem" by Van Vleck.

Theorem. The orthonormality relation

$$\sum_R \sqrt{\frac{l_j}{h}}\, \Gamma^{j}_{\mu\nu}(R) \sqrt{\frac{l_{j'}}{h}}\, \Gamma^{j'}_{\nu'\mu'}(R^{-1}) = \delta_{jj'}\delta_{\mu\mu'}\delta_{\nu\nu'}$$

is obeyed for all the ~~(inequivalent)~~ irreducible representations of a group, where the summation is over all h group elements $A_1, A_2, \dots, A_h$ and $l_j, l_{j'}$ are, respectively the dimensionalities of the representations Γ^j and $\Gamma^{j'}$.
If the representations are unitary, the orthonormality relation becomes

$$\sum_R \sqrt{\frac{l_j}{h}}\, \Gamma^{j}_{\mu\nu}(R) \sqrt{\frac{l_{j'}}{h}}\, \Gamma^{j'}(R)^{\dagger}_{\nu'\mu'} = \delta_{jj'}\delta_{\nu\nu'}\delta_{\mu\mu'}$$

proof: Consider an arbitrary matrix X with $l_{j'}$ rows and l_j columns and construct from it the matrix M:

$$M = \sum_R \Gamma^{j'}(R)\, X\, \Gamma^{j}(R^{-1}) \qquad \dim|M| = l_{j'} \times l_j$$

Multiply M by $\Gamma^{j'}(S)$ to the left.

$$\Gamma^{j'}(S)M = \Gamma^{j'}(S)\sum_R \Gamma^{j'}(R)X\Gamma^{j}(R^{-1}) =$$

$$= \sum_R \Gamma^{j'}(SR)\, X\, \Gamma^{j}(R^{-1}S^{-1})\,\Gamma^{j}(S) \quad = \text{ by rearrangement theorem}$$

$$= \sum_R \Gamma^{j'}(R)\, X\, \Gamma^{j}(R^{-1})\,\Gamma^{j}(S) = M\,\Gamma^{j}(S)$$

Case 1 $\quad \ell_j \neq \ell_{j'}$ or $\ell_j = \ell_{j'}$ but Γ^j is inequivalent to $\Gamma^{j'}$

It follows from SCHUR'S LEMMA (part 2) that $M = 0$. If we take $X: X_{\gamma\lambda} = \delta_{\gamma\nu}\delta_{\lambda\nu'}$

$$0 = M_{\mu\mu'} = \sum_R \sum_{\gamma\lambda} \Gamma^{j'}_{\mu\gamma}(R)\delta_{\gamma\nu}\,\delta_{\lambda\nu'}\,\Gamma^{j}_{\lambda\mu'}(R^{-1}) =$$

$$= \sum_R \Gamma^{j'}_{\mu\nu}(R)\,\Gamma^{j}_{\nu'\mu'}(R^{-1}) \qquad \forall\ \mu,\mu' \quad \nu,\nu'$$

Case 2 $\quad \ell_j = \ell_{j'}$ and the two representations are equivalent $\Rightarrow$ Schur's lemma (part 1) tells us that $M = c\mathbb{1}$

$$M_{\mu\mu'} = c\,\delta_{\mu\mu'} = \sum_R \sum_{\lambda\gamma} \Gamma^{j'}_{\mu\gamma}(R)\ X_{\gamma\lambda}\ \Gamma^{j'}_{\lambda\mu'}(R^{-1})$$

$$X_{\gamma\lambda} = \delta_{\gamma\nu}\delta_{\lambda\nu'}$$

$$\Rightarrow \quad c_{\nu\nu'}\delta_{\mu\mu'} = \sum_R \Gamma^{j'}_{\mu\nu}(R)\,\Gamma^{j'}_{\nu'\mu'}(R^{-1}) \quad (*)$$

$c_{\nu\nu'}$ since c depends on the particular choice of X.

$\mu = \mu'$ and sum over μ

$$\ell_j c_{\nu\nu'} = \sum_\mu \sum_R \Gamma^{j'}_{\mu\nu}(R)\Gamma^{j'}_{\nu'\mu}(R^{-1}) = \sum_R \Gamma^{j'}_{\nu\nu'}(E) = h\,\delta_{\nu\nu'}$$

$$c_{\nu\nu'} = \frac{h}{\ell_{j'}}\delta_{\nu\nu'}$$

(*) That is $\quad \displaystyle\sum_R \Gamma^{j'}_{\mu\nu}(R)\,\Gamma^{j'}_{\nu'\mu'}(R^{-1}) = \frac{h}{\ell_{j'}}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}$

Putting all together:

$$\sum_R \sqrt{\frac{\ell_j}{h}}\,\Gamma^{j}_{\mu\nu}(R)\,\sqrt{\frac{\ell_{j'}}{h}}\,\Gamma^{j'}_{\nu'\mu'}(R^{-1}) = \delta_{jj'}\,\delta_{\mu\mu'}\,\delta_{\nu\nu'}$$

■