University of Regensburg — Summer Term 2014

Applications of Group Theory

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Lectures — 9.2.01, Mondays, 14:15

Exercises — H34, Wednesdays, 14:15

Sheet 2

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1. Groups and subgroups

1. Show that the set $\{1, -1, i, -i\}$ (where i is the imaginary unit) with the complex multiplication as binary composition, is a group $\mathcal{G}$.
2. A subset $\mathcal{H}$ of $\mathcal{G}$, that is itself a group with the same law of binary composition, is a subgroup of $\mathcal{G}$. That is, $\mathcal{H}$ has to satisfy closure as all other properties are automatically fulfilled. Find all the subgroups of $\mathcal{G}$.

2. Conservation of the norm

Consider a vectorial space V on which a scalar product is defined as a bilinear function by the relation $<e_i, e_j>= \delta_{ij}$, where δ_{ij} is the Kronecker function, and $i(j) = 1, \ldots, n$ labels the n elements $e_{i(j)}$ of a complete basis for V. Prove that each linear transformation f in V which conserves the scalar product between vectors, *i.e.* $<f(v), f(w)>=<v, w>$ is represented by a unitary matrix *i.e.* $M_f M_f^\dagger = M_f^\dagger M_f = \mathbf{1}$, where $\mathbf{1}$ represents the identity matrix.

3. Matrix representations

In the lecture we have introduced the set of homomorphisms connecting point groups to groups of 3x3 matrices generating linear mappings of $\mathbb{R}^3$ into itself. Moreover we related the latter to a group of function transformations which can eventually be mapped back into matrix groups once a vectorial space of functions invariant under the group of functional operations is introduced. Let us now make a concrete example:

1. Construct the matrix that generates, in $\mathbb{R}^3$, C_4^+, *i.e.* the anticlockwise rotation of $\pi/2$ with respect of the z axis.
2. Construct the associated function operator $\hat{R}_{C_4^+}$ and find the transformed function for each of the 3 hydrogen $2p$ orbitals. Find the associated matrix representation of the point symmetry operation in the Hilbert space generated by such orbitals.
3. Repeat the first two steps for all the elements of the cyclic group C_4. You have obtained a matrix representation of the group.
4. Analogously, find the matrix representation of the dihedral group D_4 in the same Hilbert space.

Frohes Schaffen!