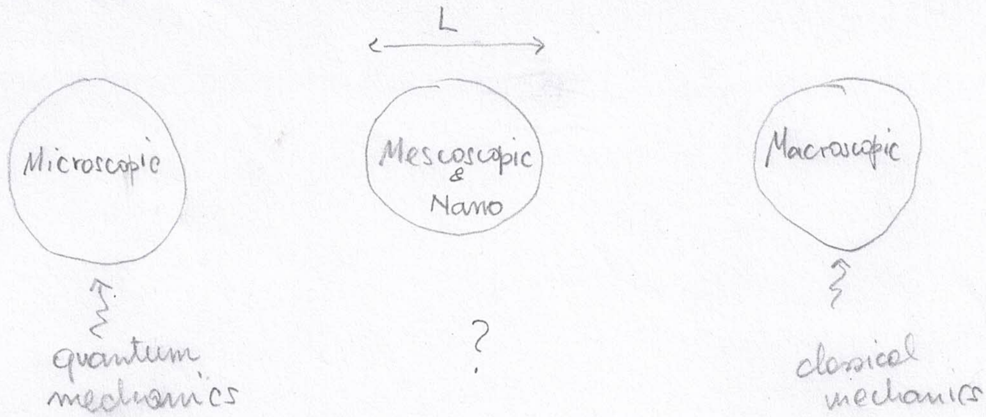
Chapter 1 Preliminary concepts cf. Ch.1 Datta

Meso = greek word for "middle" $1\mu m = 10^{-6} m$

Nano = greek word for "dwarf" $1\, nm = 10^{-9} m$

It involves physical phenomena which occur in physical systems in the middle between the <u>microscopic</u> and the <u>macroscopic</u> world:



In practice, microscopic systems have sizes $\lesssim 1 \AA$

classical systems are visible by eye.

- In this course we shall investigate how fundamentally <u>quantum</u> phenomena impact the <u>transport</u> characteristics of "small" conductors.

  Those are:

  - Energy discretization, charge quantization $\leftrightarrow$ reduced dimensionality
  - Quantum tunneling, interference $\longleftrightarrow$ particle-wave duality
  - Collective phenomena (e.g. superconductivity) $\leftrightarrow$ phase coherence

- Mesoscopic phenomena occur only if the system is "small enough". On the other hand the overall system is still "large enough" that several thousands of atoms are involved in some parts of the device.

To this extent our mesoscopic device must have a size L smaller than (at least one) of the characteristic scales governing some quantum phenomena.

The most relevant ones will be:

- l_e (elastic) mean free path
- λ_F Fermi wave length
- l_φ phase coherence length
- l_i inelastic scattering length

Specifically:

l_e Consider a metallic conductor, say a wire. Classically, the resistance R depends linearly on the wire length L:

$$R = \frac{L}{\sigma A} \qquad (1.1)$$

where σ = conductivity, A = Area of a cross-section.

The reason is that a metal crystal is never perfect, and scattering by lattice distortions (phonons) or defects impact the electronic transport

l_e is the average distance one electron travels between scattering events due to impurities

A microscopic calculation due to Drude yields

$$\sigma_D = \frac{n e^2 \tau}{m}, \qquad \tau = l_e / v_F \qquad (1.2)$$

With n = electron density, v_F = Fermi velocity.

It is valid when $L \gg l_e$ (diffusive regime).
It also implies Ohm's law

$$I = \frac{V}{R} \qquad (1.3)$$

Stating that a direct proportionality exists between the current I flowing through the wire and the applied bias voltage V.

Let us now imagine that our wire length is

$$L < l_{el} \quad \text{ballistic regime}$$

We would expect by applying (1.2) that $\sigma_D \sim \tau \sim \infty$
Nevertheless, we shall see that, due to quantum confinement, the resistance of the wire is finite, and in some cases is quantized in a peculiar way.

$\underline{\lambda_F}$ It is the scale associated to electrons with momentum close to the Fermi momentum

$$\lambda_F = \frac{2\pi}{k_F}, \quad k_F \text{ Fermi momentum}$$

For 3d electrons $(n \to n_3)$

$$k_F = (3\pi^2 n_3)^{1/3}$$

n_d = electronic density in d-dimen

$n_3 = N/V$, $n_2 = \frac{N}{A}$

For 2d electrons

$$k_F = (2\pi n_2)^{1/2}$$

E.g. $n_2 = 5 \cdot 10^{11}\ cm^{-2} \to \lambda_F \sim 35\ nm$

l_φ It is the scale over which coherence effects can be seen. E.g. interference between electron paths in electron interferometers.

Note: for weak disordered systems $l_\varphi \sim \sqrt{D\tau_\varphi}$ cf. later, D diffusion coeff
for quasi-ballistic systems $l_\varphi \sim l_i$

l_i Length scale for inelastic processes, e.g. due to phonons or photons (leading to relaxation and equilibration)

Notice: only inelastic scatterers, not static impurities, destroy phase coherence. In this case $l_\varphi \sim l_i$

Orders of magnitude: how large are the various lengths are depends on the studied system.
In general though, it holds

bulk metals: $\lambda_F \ll \{l_{ee}, l_i\}$

mesoscopic systems: $\lambda_F \lesssim l_{ee} < l_\varphi \sim l_i$

in particular, in metals is $\lambda_F \sim 0.1 - 1$ nm
2DEG is $\lambda_F \sim 10 - 100$ nm

and in metals is $l_{ee} \sim 10 - 100$ nm
2DEG is $l_{ee} \sim 100$ µm

moreover l_i and l_φ depend on temperature
(the impact of phonons, being lattice vibrations, decreases with temperature)

l_φ It is the scale over which quantum coherence effects can be seen, e.g. interference between electron paths in electron interferometers

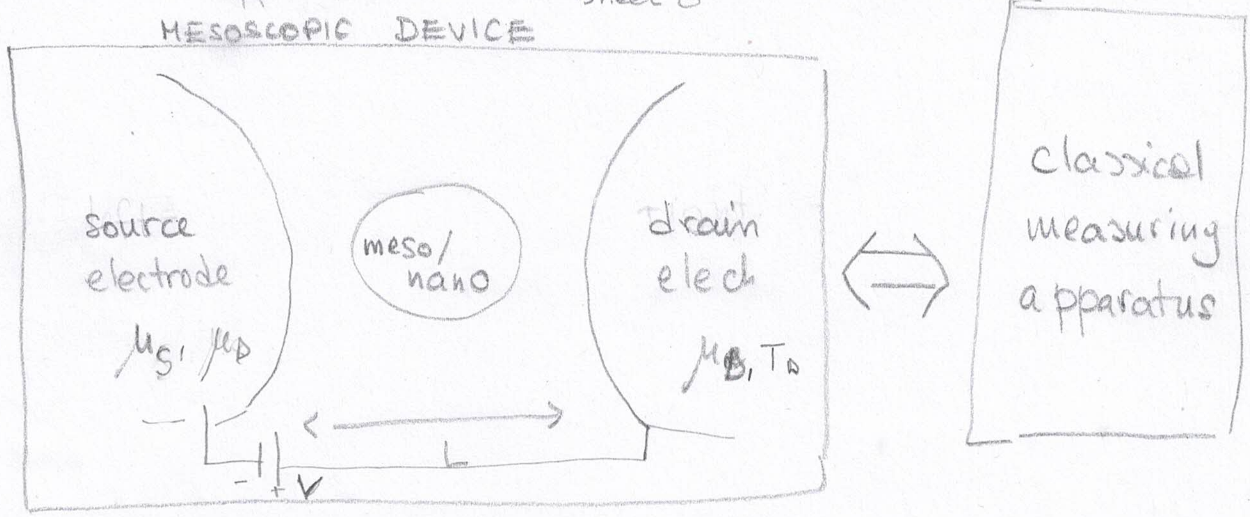
l_i length scale for inelastic scattering e.g. due to phonons or photons

Notice: only inelastic scatters, not impurities, destroy phase coherence

1.2 Generic transport set-up

Even if we shall consider various kinds of mesoscopic & nanoscopic devices showing different types of transport properties & phenomena, the set-up we shall look at is generic:

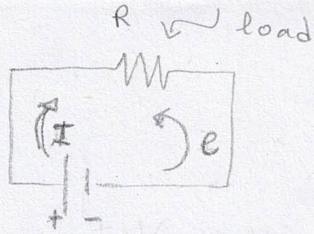
a small conductor conductor contacted to <u>electrodes</u> which in turn are coupled to a macroscopic external apparatus (⇝ see examples sheet ○)



- the electrodes ideally work as a reservoir (heat and particle bath → grand canonical ensemble)

→ once electrons enter the reservoir, they quickly thermalize with the lattice, the internal state of the reservoir does not change considerably

Note: classical circuit



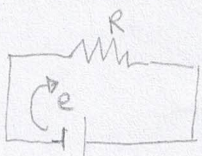
$I = \Delta V / R$ Ohm's law

$\Delta V = (V_+ - V_-) = V_L - V_R$

battery
(acts as reservoir of electrons)

Introduce electrochemical potentials $\mu_L = eV_L$, $\mu_R = eV_R$, $e<0$

Consider $\mu_L - \mu_R > 0 \leftrightarrow V_L - V_R = \Delta V < 0$



$\mu_L - \mu_R > 0$

electron flow from larger to smaller electrochemical potential

Example of mesoscopic/nanoscopic devices

i) Quantum point contact

ii) Quantum dots

iii) Single atom and single molecule junctions

iv) Exercise sheet 0 with other examples

Note: top down & bottom up approaches

top-down :- laterally defined heterostructures (gate defines on top of 2DEG)
- etched 2D materials as graphene

bottom up :- single-molecule based junctions or
self-grown nanowires e.g. carbon nanotubes

; Electrons in the metallic reservoirs are noninteracting fermions

→ particles' distribution function has Fermi-Dirac form:

$$f_{S/D}(E) = \frac{1}{\exp\left[(E-\mu_{S/D})/k_B T_{S/D}\right]+1} \qquad (1.4)$$

k_B Boltzmann constant

T lattice temperature

Note: Because the classical measurement apparatus works at room temperature, care has to be taken that the temperature of the meso/nano device is at the low temperatures necessary to see mesoscopic effects.

Note: While the reservoirs can be described using Eq. (1.4) this is no longer the case for the central system. Which kind of (nonequilibrium) distribution exists, depends on the length scales in the system, its dimensionality, and its topology (e.g. ring vs wire)

Three cases can be distinguished:

regime	lengths	conserved quantities
ballistic	$L \ll l_{ee}, l_i$	momentum $\vec{p}$ → momentum resolved current energy E, charge q
diffusive nonequilibrium	$l_{ee} \ll L \ll l_i$ (elastic scattering randomizes momentum)	energy E & charge q → E resolved contribution to current
local equilibrium	$l_{ee}, l_i \ll L$ (e-ph scattering allows equilibration)	charge q → conventional charge current

Quantum point contacts

Quantized Conductance of Point Contacts in a Two-Dimensional Electron Gas

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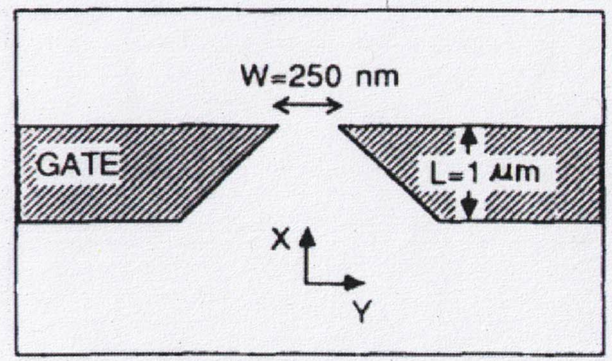
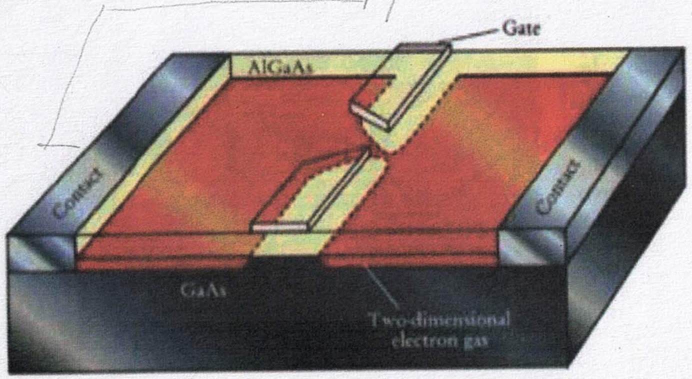
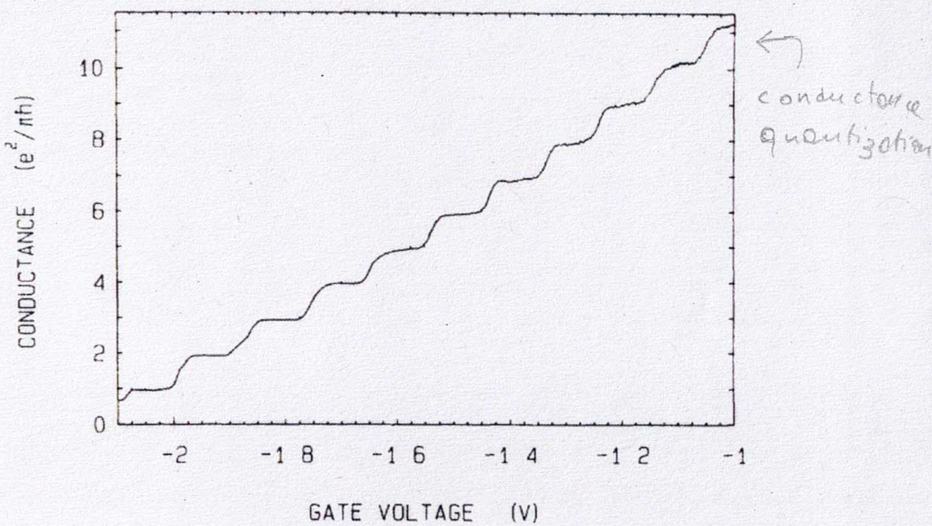
(Received 31 December 1987)

Ballistic point contacts, defined in the two-dimensional electron gas of a GaAs-AlGaAs heterostructure, have been studied in zero magnetic field. The conductance changes in quantized steps of $e^2/\pi\hbar$ when the width, controlled by a gate on top of the heterojunction, is varied. Up to sixteen steps are observed when the point contact is widened from 0 to 360 nm. An explanation is proposed, which assumes quantized transverse momentum in the point-contact region.







van Wees at al., PRL **60**, 848 (1988)

Quantum dots

Coulomb Blockade of Single-Electron Tunneling, and Coherent Oscillations in Small Tunnel Junctions

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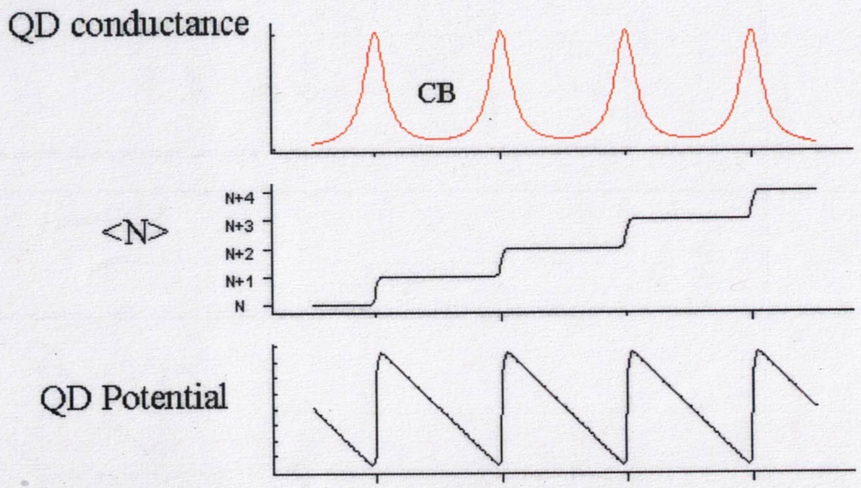
Department of Physics, Moscow State University, Moscow USSR

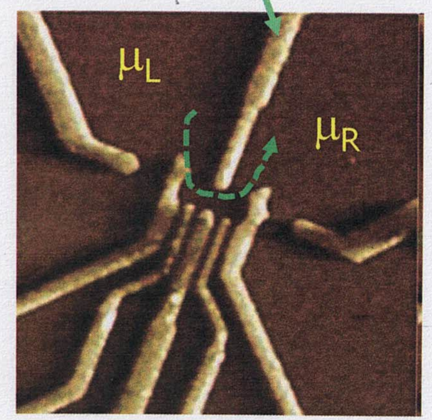
(Received September 9, 1985)

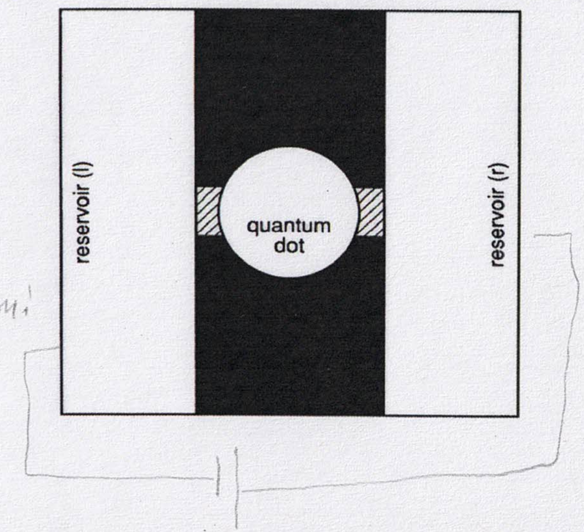
A microscopic approach to the theory of small, current-biased tunnel junctions is developed. This approach yields a natural account of the "secondary" quantization of both the single-electron (quasiparticle) and Cooper-pair (Josephson) current components. The theory shows that the current of the single electrons is blocked by their Coulomb interaction at low temperatures within a considerable range of the junction voltage. As a result of the blockade, coherent oscillations of the voltage can arise even in the absence of Josephson coupling, e.g., for single-electron tunneling (SET) between normal metal electrodes. The most significant features of these "SET" oscillations and their coexistence with Bloch oscillations in Josephson junctions are studied in detail. Prospects of experimental verification of the predicted effects and of their possible applications are discussed.



← charge quantization!
⇓
artificial atom

plunger gate

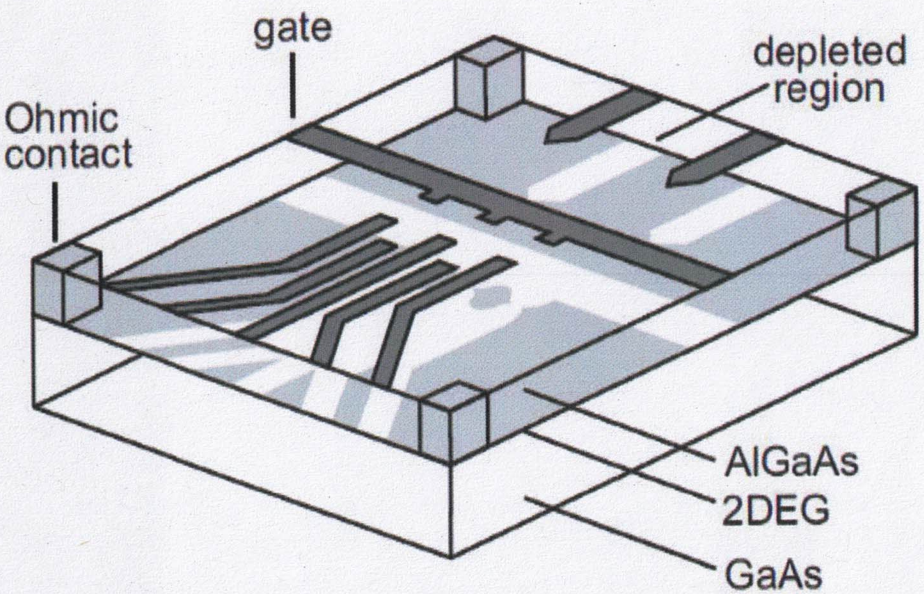




Averin and Likharev, J. of Low Temp. Phys. **62**, 345 (1986)

Quantum dots

a



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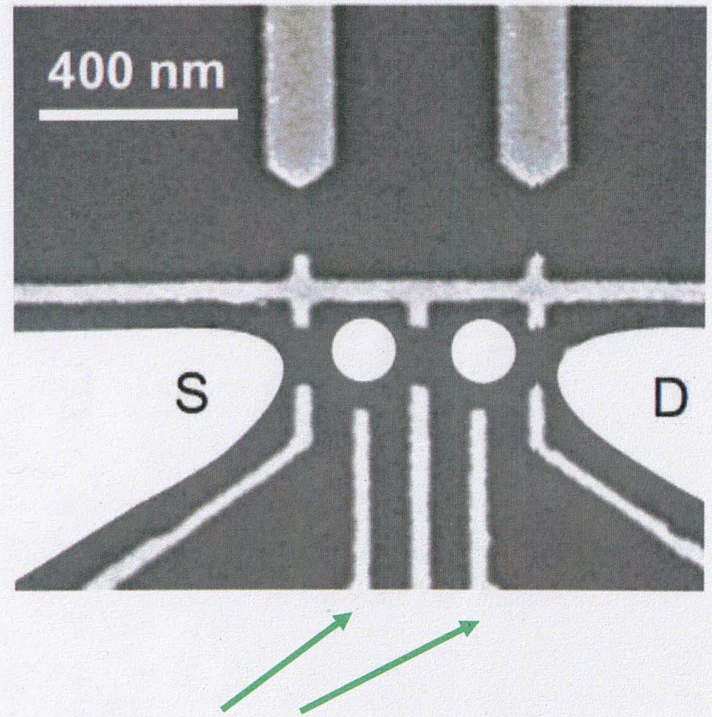
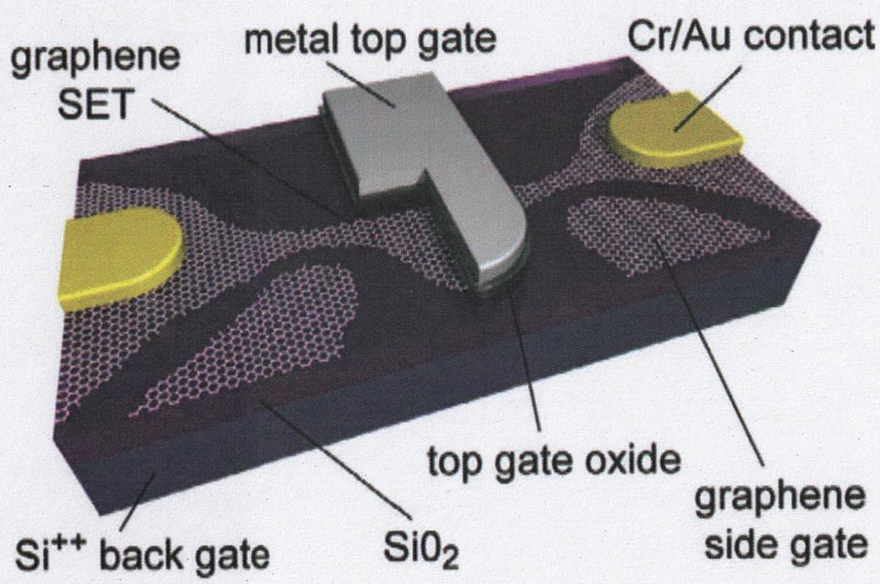
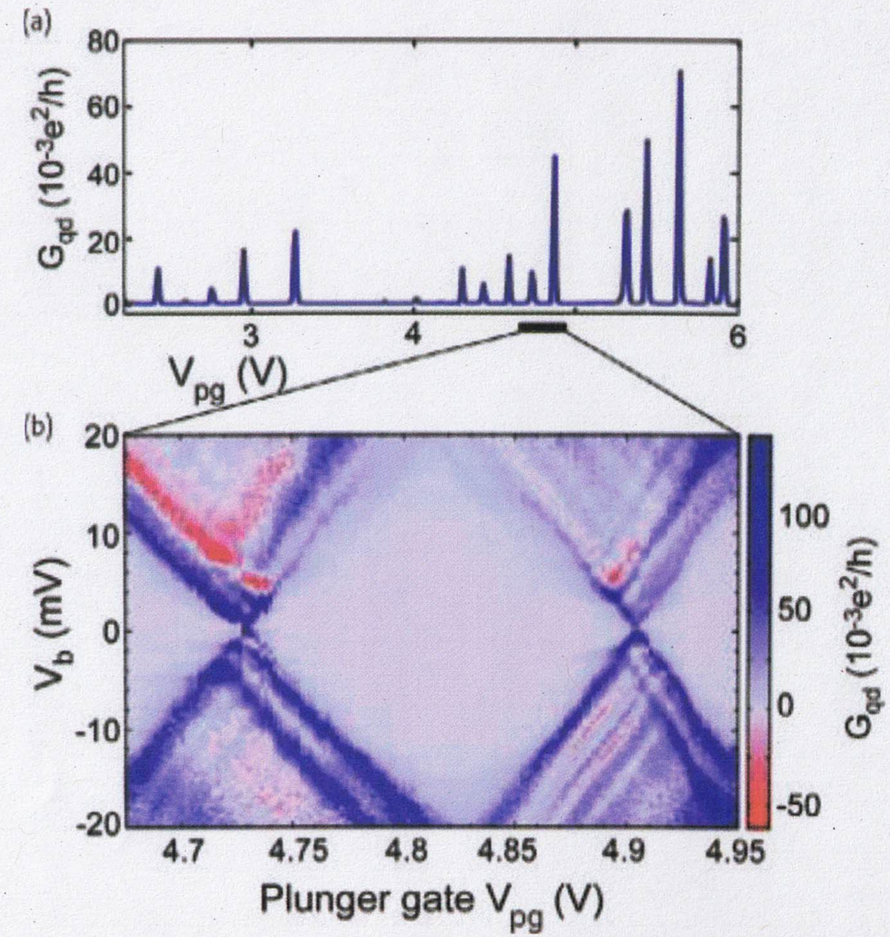


Kouwenhoven et al., Rep. Prog. Phys. **64** 701 (2001)

- quantized electron number + tunneling between dots
  ⇓
  artificial molecule
- electrons: charge & spin
  ⇓
  spins degree of freedom as quantum bits (qubits)

Graphene quantum dots

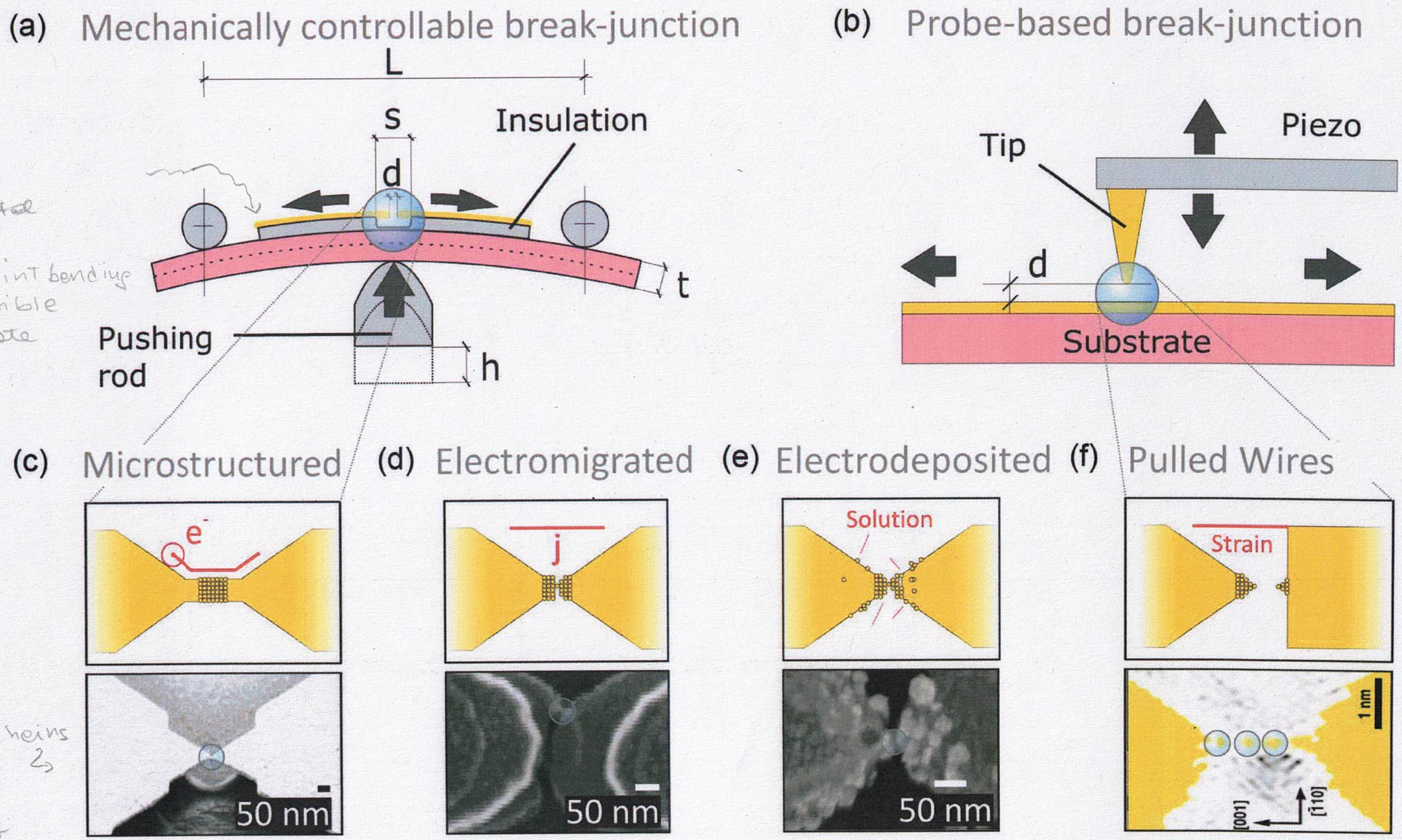




- contacts can be of very different materials!
  - ferromagnets => spintronics
  - superconductors => Josephson currents

Ihn et al., Materials Today **13**, 44 (2010)

Single atom & molecule junctions



thin metal

• three-point bending of a flexible substrate

- atomic chains
- single atom contact

• feed back mechanism allows to control distances with sub nm precision

• possibility of imaging

Schwarz and Lörtscher, J. Phys.: Condens. Matter **26,** 474201 (2014)

What else should we care about?

- dimensionality (2D, 1D, 0D)
- topology & geometry (wire, ring, quantum point contact, ...)
- how is the system contacted to the reservoir

  (tunnel contact vs electron wave guide)

1.3. Size quantization

e.g. chapter 1.6 of Datta
chapter 2.3 Bruus & Flensberg

In the lecture we want to consider systems where at least one length scale among

$$w = \text{width} \;, \quad d = \text{thickness} \;, \quad L = \text{length}$$

is "small enough" that the systems does not behave as a three-dimensional conductor (3D), but rather as a two-dimensional (2D), one-dimensional (1D) or even zero-dimensional (0D) one.

To understand this concept, imagine that we can neglect e-e interactions, such that the time-evolution of a conduction electron is fully governed by the single-particle Schrödinger equation

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + U(\vec{r}) + V_{latt}(\vec{r}) \right] \psi(\vec{r}) = E\, \psi(\vec{r}) \quad (1.5)$$

where $U(\vec{r})$ = confinement potential

$V_{latt}(\vec{r}) = V_{latt}(\vec{r}+\vec{R}) \quad \forall \vec{R}$ = periodic potential due to ionic lattice, $R \in$ Bravais lattice

Using Pauli principle, the allowed single particle states are filled by one electron only (including spin) up to the Fermi level E_F.

- if $U = 0$ $\Rightarrow \psi_{\vec{k}}(\vec{r}) = u_{\vec{k}}(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}$, $u_{\vec{k}}(\vec{r}+\vec{R}) = u_{\vec{k}}(\vec{r})$ Bloch waves

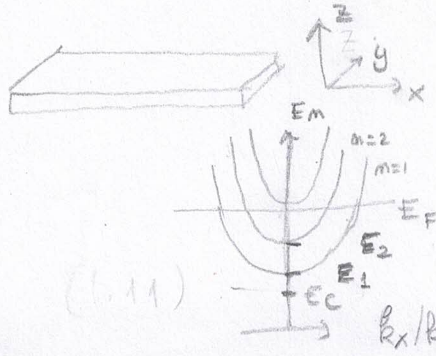
- if $V_{latt} = 0$, $\begin{cases} U = 0 \Rightarrow \psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} & \text{plane waves} \\ U \neq 0 \Rightarrow & \text{subbands} \end{cases}$

this approximation is actually valid in semiconductors where, in the vicinity of the bottom of the conduction band, the effect of the lattice potential V_{latt} is encapsulated in an effective mass m^* and the energy is evaluated from the bottom E_c of the band. Thus

$$(1.5) \xrightarrow[\text{(there } m = m^*)]{\text{semiconductors}} \Big[\underbrace{-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{r}) + E_c}_{\hat{H}}\Big]\psi(\vec{r}) = E\psi \quad (1.6)$$

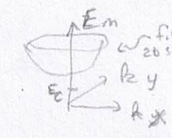
Consider now

i) $\underline{U(\vec{r}) = U(z)}$ $\longrightarrow$ 2D subbands , 2d systems

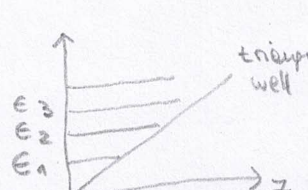


$$\psi(\vec{r}) = \phi_m(z)\, e^{ik_x x}\, e^{ik_y y}$$

$$\hookrightarrow E_m(k_x, k_y) = E_c + \underset{\substack{\uparrow \\ \text{transverse subband energy}}}{E_m} + \frac{\hbar^2}{2m}(k_x^2 + k_y^2) \quad (1.7)$$



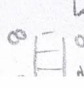
e.g. $U(z)$



2D see exercise sheet 2

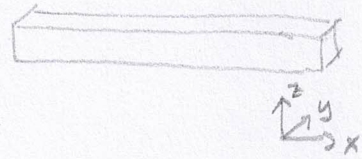
if $E_2 > E_F$ the system stays in the first subband and it behaves effectively as a 2D system

$U(z)$



square well $E_m \sim \frac{\hbar^2}{2m}\left(\frac{m\pi}{d}\right)^2$

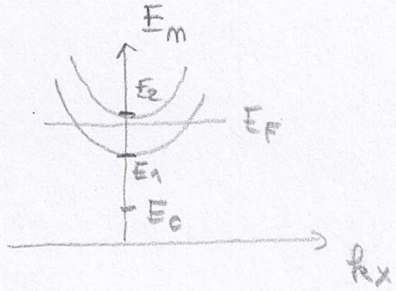
ii) $\underline{U(\vec{r}) = U(y,z)}$ $\longrightarrow$ 1d subbands , 1d system



$$\psi(\vec{r}) = \phi_m(y,z)\, e^{ik_x x}$$

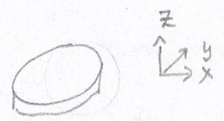
$$E_m(k_x) = E_c + \epsilon_m + \frac{\hbar^2}{2m} k_x^2 \qquad (1.8)$$

$\uparrow$ transverse subband



if $E_2 > E_F$ the system is effectively 1D

iii) $U(\vec{r}) = U(x,y,z)$ $\longrightarrow$ 0d states, confinement in all directions!



Note:

From the above considerations it is clear that E_F determines (and hence the electronic density) the dimensionality of the system. Notice that if a few low subbands are involved in transport, one says that the system is "quasi-two dimensional" or "quasi-one dimensional"

1.4 Fermi level

It is thus important to be able to determine the Fermi level of a generic mesoscopic system. Given the set $\{E_i\}$ of eigenstates of the Hamiltonian $\hat{\mathcal{H}}$, the Fermi level is defined by the condition that the N (conduction) electrons occupy the lowest levels without leaving voids:

$$N = \sum_i f(E_i) \overset{T=0}{=} \sum_i \Theta(E_F - E_i) \qquad (1.9)$$

Fermi function

index i is a collective index e.g. $i = (\vec{k}, \sigma)$

It is thus convenient to introduce the concept of density of states:

$$D(E) \equiv \sum_i \delta(E - E_i) \qquad (1.10) \qquad \text{(DOE)}$$

yielding

$$N = \int dE \; D(E) f(E) \qquad (1.11)$$

It is usually easier to calculate electronic properties, including E

once the DOE is known ($\rightarrow$ cf. Exercise 1 sheet 2)

For the example of 2D-confinement:

$$D(E) = \sum_{m,\sigma} \sum_{k_x k_y} \delta(E - E_m(k_x k_y)) = \sum_{m,\sigma} \sum_{m_x m_y} \delta(E - E_m(k_x, k_y)) \qquad (1.12)$$

$i = (k_x, k_y, m, \sigma)$

periodic boundary conditions along $x-y$

$$k_x = m_x \frac{2\pi}{L_x}, \quad \Delta k_x = \frac{2\pi}{L_x}$$

$$k_y = m_y \frac{2\pi}{L_y}, \quad \Delta k_y = \frac{2\pi}{L_y}$$

Density of states of a (quasi)-2D system

From (1.10)

$$D(E) = \sum_{m,\sigma} D_m(E) \qquad \text{where} \quad D_m(E) = \sum_{k_x k_y} \delta(E - E_m(k_x,k_y)) \quad (1.13)$$

Using $E_m = E_c + \varepsilon_m + \frac{\hbar^2}{2m}(k_x^2+k_y^2) = E_c + \varepsilon_m + \frac{\hbar^2}{2m}k^2 = E_m(k)$,

it follows from the quantization of k_x, k_y in the continuum limit

$$D_m(E) = \frac{L_x L_y}{(2\pi)^2} \int dk_x \int dk_y \; \delta(E - E_m(k))$$

$$= \frac{L_x L_y}{(2\pi)^2} \int_0^\infty 2\pi k \, dk \; \delta(E - E_m(k))$$

$$= \frac{L_x L_y}{(2\pi)^2} \int_0^\infty \frac{m}{\hbar^2} 2\pi \, d\varepsilon \; \delta(E - E_c - \varepsilon_m - \varepsilon)$$

$\varepsilon(k) = \frac{\hbar^2 k^2}{2m} \longrightarrow \frac{d\varepsilon}{dk} = \frac{\hbar^2 k}{m}$, $\quad d\varepsilon = \frac{\hbar^2 k}{m} dk$

$$\Rightarrow D_m(E) = \frac{L_x L_y}{2\pi} \frac{m}{\hbar^2} \, \theta(E - E_c - \varepsilon_m) \qquad (1.14)$$

$\int_0^\infty \delta(\varepsilon - E_x)\, d\varepsilon = 1$ only if $E_x > 0$

$D_m(E)$ is constant for a 2D system! (it diverges in thermodynamic limit $A = L_x L_y \to \infty$; $\frac{D_m(E)}{A} \equiv d_m(E)$ is finite)

Fermi energy of a (quasi)-2D system

$$N = \int dE\, D(E) f(E) \underset{T=0}{=} \int dE\, D(E)\, \theta(E_f - E) = \sum_{m,\sigma} \int dE\, D_m(E)\, \theta(E_f - E)$$

$$= \underset{\text{spin}}{2} \frac{L_x L_y\, m}{2\pi \hbar^2} \sum_m \int_{E_c+\varepsilon_m}^{E_F} dE = \frac{L_x L_y\, m}{\pi \hbar^2} \sum_{m}^{m_c} (E_F - E_c - \varepsilon_m) \qquad (1.15)$$

for all $m \le m_c$ such that $E_F - E_c - \varepsilon_m > 0$

From (1.15)

$$n_2 = \frac{N}{L_x L_y} = \frac{m}{\pi\hbar^2} \sum_{n}^{n_c} (E_F - E_c - \varepsilon_n)$$ ← for all $n \leq n_c$ such that the quantity in ( ) stays positive

i.e. there are various Fermi momenta

↳ if $E_c + \varepsilon_2 > E_F > E_c + \varepsilon_1$ (one subband only)

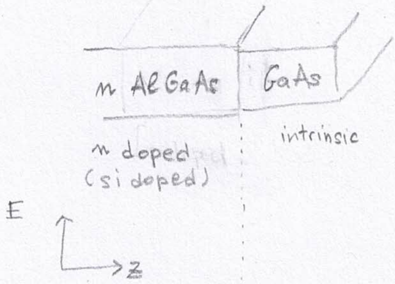
$$n_2 = \frac{m}{\pi\hbar^2}\Big[\underbrace{E_F - E_c - \varepsilon_1}_{\tilde{E}_F}\Big] \equiv \frac{m}{\pi\hbar^2}\tilde{E}_F$$ ← energies measured from bottom of the first subband

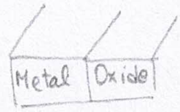
$$\Rightarrow n_2 = \frac{m}{\pi\hbar^2}\frac{\hbar^2 k_F^2}{2m} \Rightarrow \boxed{k_F^{2D} = \sqrt{2\pi n_2}} \quad (1.16)$$

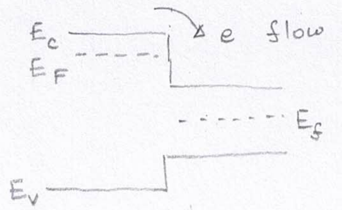
Hystorically, the first low dimensional system that has been used.

- Metal Oxide Semiconductor field effect transistor
- GaAs/AlGaAs Heterostructures

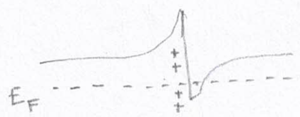
The main idea is the same







e flow after the two materials are brought together → positive charge left behind
↓
electrostatic potential causes band bending

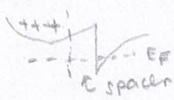


- negative charge accumulates in the potential well
- confinement along z causes level quantization

$\psi(\vec{r}) = \varphi_m(z)\,\chi(x,y)$ ← plane wave in x-y plane

↑ nth transverse channel ↔ subband

Note: in reality a spacer is added between AlGaAs and GaAs to ensure distance between Si donors and electrons



$\hat{H}\psi = E\psi \Rightarrow E_n = E_c + \mathcal{E}_m + \frac{\hbar^2}{2m}(k_x^2 + k_y^2)$ (1.7)

↑ bottom of the band

advantages:

- very high carrier density n_2 (2DEG is very thin)
- very high mobility $\mu^{(*)}$ (donor impurities are far inside n-AlGaAs) in the presence of a spacer
- in situ controll of n_2 possible via metal gates on top of the heterostructure

↳ dimensionality can be further reduced: quantum wires, quantum dots

(*) note: mobility

$\longrightarrow e\vec{E}$

In the presence of an electric field, the electrons acquire a drift velocity $\vec{v}_d$ superposed to their random motion

$$\mu := \frac{|\vec{v}_d|}{|\vec{E}|} \quad (1.8)$$

1989 $\mu \sim 10^7\ cm^2/Vs$
Pfeiffer et al. APL 55 1888 (19..
nowadays further improvemen...

Studied systems

- two-dimensional systems: 2DEG, graphene, ...
  (and quasi 2D) (bilayer graphene)

- one-dimensional systems: nanowires, carbon nanotubes,
  (and quasi 1D) edge states topological insulators,
  graphene nanoribbons

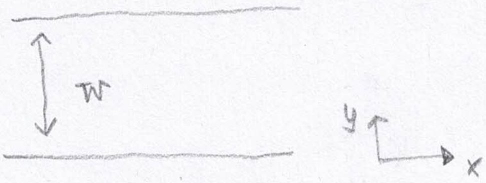
- zero dimensional systems: single-molecules,
  quantum dots

↳ exercise: magnetic subbands (effects of magnetic field ⊥ to 2D system)

1.5 Quantum point contact: constriction as a potential barrier

Ch. 2.1 Datta

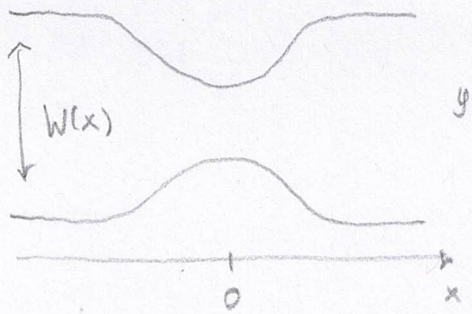
a)



$$\psi(x,y) = \varphi_m(y)\, e^{ik_x x} \rightarrow \begin{cases} E_m = \epsilon_m + \dfrac{\hbar^2 k_x^2}{2m} \\ \epsilon_m = \dfrac{\hbar^2}{2m} \dfrac{\pi m^2}{W^2} \end{cases}$$

hard wall confined (particle in a box)

b)



if variation along x smooth (adiabatic wave guide) it is still possible to make the Ansatz

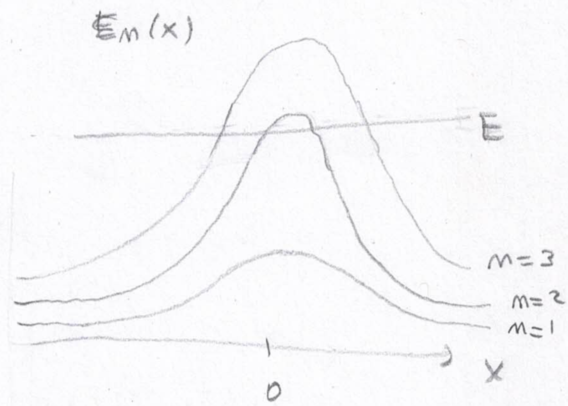
exercise sheet 3

$$\psi(x,y) = \varphi_m(x,y)\, \chi(x)$$

where now $\chi(x)$ satisfies

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \epsilon_m(x) \right) \chi(x) = E\, \chi(x)$$

and $\boxed{\epsilon_m(x) = \dfrac{\hbar^2 \pi^2}{2m} \dfrac{m^2}{W^2(x)}}$ (1.17)



$\Rightarrow$ the constriction acts as a barrier,

The larger the mode the higher the barrier!

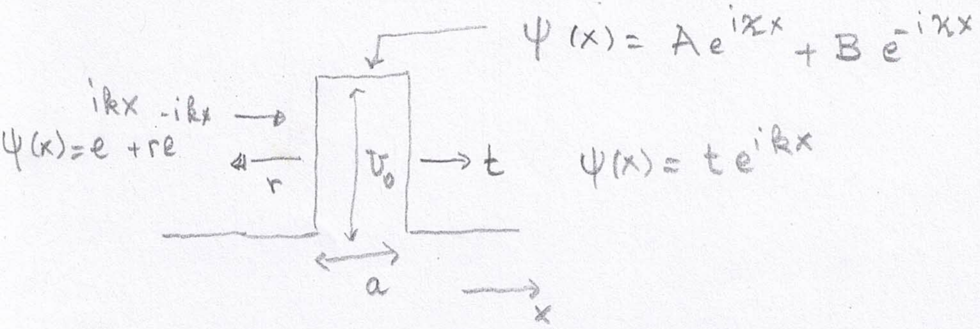
- Imagine now that to each mode it corresponds an impenetrable barrier $\Rightarrow$ an electron impinging at it is either reflected or transmitted depending on whether the electron has energy $E \gtrless E_m(0)$ ($E_m(x=0)$ mode cut-off)

$\Rightarrow$ there are only a finite number of open channels

(all the other are closed)

In reality, the barrier is not impenetrable, but it has an energy-dependent <u>transmission coefficient</u> $T_n(E)$.

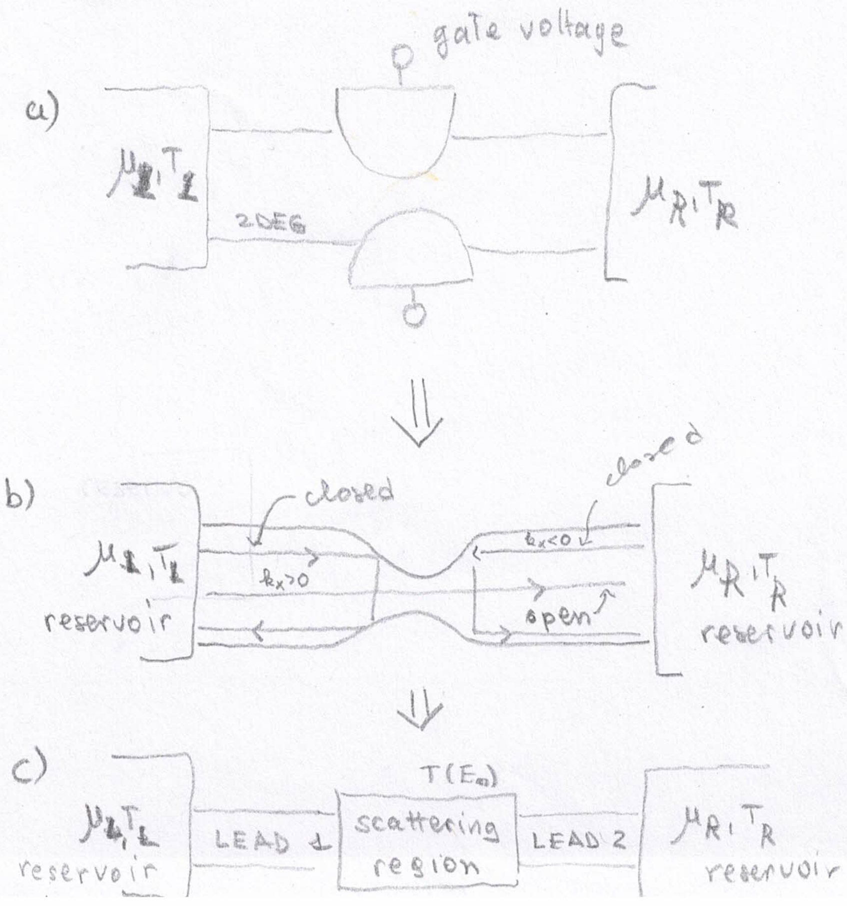
E.g. transmission through a rectangular barrier:



$$T(E) \equiv |t|^2 = \frac{1}{1 + \left(sh^2 \varkappa a \underbrace{\frac{(\varkappa^2 + k^2)}{2k\varkappa}}_{\epsilon/2} \right)^2} \qquad (1,18)$$

with $E = \frac{\hbar^2 k^2}{2m}$, $V_0 - E = \frac{\hbar^2 \varkappa^2}{2m}$ $\rightarrow \frac{\epsilon^2}{4} = \frac{1}{4} \frac{V_0}{E(V_0 - E)}$

Current through a QPC



- high mobility 2DEG → ballistic motion
- a negative gate voltage on a pair of metallic gates is used to deplete and narrow down the constriction ("split gate configuration)
- electrons with energy E leave/enter reservoirs

<u>assumption</u>: "reflectionless reservoirs": the electrons enter them without suffering reflections

- close & open channels
- transport ↔ scattering problem

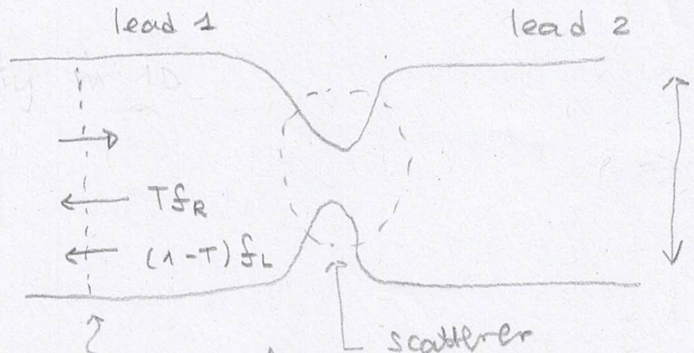
As we are going to discuss later, for a ballistic conductor like the QPC, we can tranform the transport problem in a scattering problem.

For the time being we observe that:

- the current is conserved through any section of the wire
- there are two contributions to the current:

i) plane waves with momentum $k_x > 0$ ($k_x < 0$) ejected from left (right) reservoir with weight set by $f_L(E_m)$, $f_R(E_m)$

ii) waves are reflected (transmitted) with weight $1-T$ (weight



T = average probability of being transmitted for given energy

group velocity

$$v(E_m) = \frac{1}{\hbar}\frac{\partial E_m(\varepsilon)}{\partial k_x} \Rightarrow \text{at section} \quad v\left[f_L - T f_R - (1-T) f_L\right]$$

$$= vT\left[f_L - f_R\right]$$

$$\begin{cases} E_m = E_c + \varepsilon_m + \varepsilon(k_x) & \underline{\text{far from barrier}} \\ \varepsilon_m = \frac{\hbar\pi^2}{2m}\frac{n^2}{W} = \varepsilon_m(-\infty) = \varepsilon_m(+\infty) \end{cases}$$

Current density

- In 1D current = current density
- Classically, $\vec{J} = ne\vec{v}$ $2 \to 1D$: $n = n_1 = \frac{N}{L}$, $\vec{v}$ = drift velocity
- Quantum mechanically, treating the modes as independent (no interference or scattering)

$$I = \underbrace{2}_{\text{spin}} e \sum_n \frac{1}{L} \sum_{k_x} \left[\frac{1}{\hbar} \frac{\partial E_n}{\partial k_x} T(E_n) f_L(E_n) + \frac{1}{\hbar} \frac{\partial E_n}{\partial (-k_x)} T(E_n) f_R(E_n) \right]$$

$$= 2e \sum_n \frac{1}{L} \int dE \, D_n(E) \, v(E) \, T(E) \left[f_L(E) - f_R(E) \right]$$

1D density of states

$$\begin{cases} D(E) = \sum_\sigma \sum_n \sum_{k_x} \delta(E - E_n(k_x)) = \sum_\sigma \sum_n D_n(E) \\ D_n(E) = \sum_{k_x} \delta(E - \underbrace{E_c - \mathcal{E}_n - \mathcal{E}(k_x)}_{E_n(k_x)}) \end{cases}$$

E_c also includes lowest mode energy in z direction

$$\hookrightarrow D_n(E) = \frac{L}{2\pi} \int_{-\infty}^{+\infty} dk \, \delta(E - E_n(k))$$

$$= \frac{L}{2\pi} \int_0^\infty d\mathcal{E} \, \frac{\partial k}{\partial \mathcal{E}} \, \delta(E - E_c - \mathcal{E}_n - \mathcal{E}(k))$$

change of integr. var. $d\mathcal{E} = dE_n$, $\mathcal{E} \to E_n$

$$= \frac{L}{2\pi} \int_0^\infty d\mathcal{E} \, \frac{\partial k}{\partial E_n(k)} \, \delta(E - \underbrace{E_c - \mathcal{E}_n - \mathcal{E}}_{E_n}) = \frac{L}{2\pi} \int_{\mathcal{E}_n + E_c}^\infty dE_n \frac{\partial k}{\partial E_n} \delta(E - E_n)$$

$$= \frac{L}{2\pi \hbar} \int_{\mathcal{E}_n + E_c}^\infty dE_n \, \frac{1}{v_n} \, \delta(E - E_n)$$

note: $\frac{2|e|}{h} \sim 80 \frac{nA}{meV}$

$$\Rightarrow \boxed{I = \frac{2e}{h} \sum_n \int dE \, \theta(E - \mathcal{E}_n - E_c) \, T(E) \left[f_L(E) - f_R(E) \right]} \quad (1.19)$$

Special case: $T_n(E) = \vartheta(E - E_G(0))$ impenetrable barrier (at $x=0$)

defines the height of an effective barrier at $x=0$

+ zero temperature

$$I = \frac{2e}{h} \sum_{\substack{n \text{ open} \\ (\text{for } \mu_L \geq E \geq \mu_R)}} (\mu_L - \mu_R)$$

↳ $G = \left.\frac{dI}{dV}\right|_{V=0} = \frac{2e^2}{h} M_{open}$ (1.20)

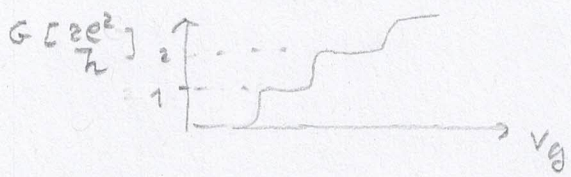
$\mu_L - \mu_R = eV$

nr. channels at the Fermi energy $\times T(E_F)$

$M_{open} = M(E_F) \times T(E_F)$

Because the number of open channels depends on the barrier height, which is controlled by V_g, the number of open channels varies with V_g

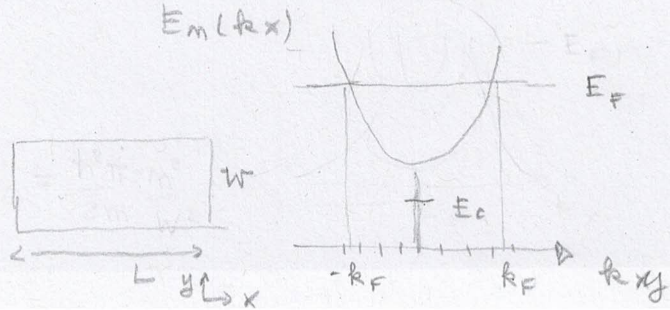
↳ First experimental observation by van Wees et al. PRL 60, 848 (1988)



Note: For generic $T(E)$ $\Rightarrow$ $\boxed{G = \frac{2e^2}{h} M T(E_F)}$ (1.21)

Note: Nr. of of modes for a wide constriction can be estimated from $\Delta k_y = \frac{2\pi}{W}$ (periodic boundary conditions)

For an electron with energy $E_F = \frac{\hbar^2 k_F^2}{2m} = \mu_0$ coming from the reservoirs, a mode can propagate if $-k_F < k_y < k_F$



$\Rightarrow M \sim \mathrm{Int}\left[\frac{2k_F}{\Delta k_y}\right] = \mathrm{Int}\left[\frac{k_F W}{\pi}\right]$

$\Rightarrow W = 15\mu$, $\lambda_F = 30$ nm $\rightarrow M \cong 1000$

Contact resistance

Note: Eq. (1.20) tells us that each transverse channel of a ballistic conductor carries a quantum of conductance $G_0 \equiv \frac{2e^2}{h}$ (including spin).

We have

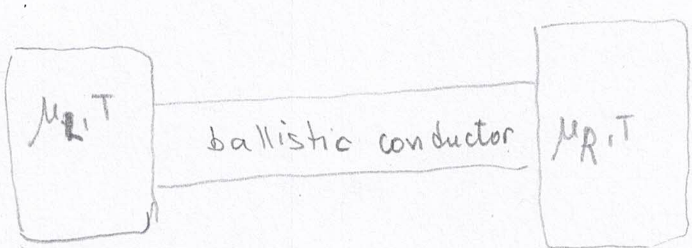
$$G_0 = \frac{1}{12.9\ k\Omega}$$

where $R_K = 12.9\ k\Omega$ is now the unit of resistance, also known as von Klitzing resistance

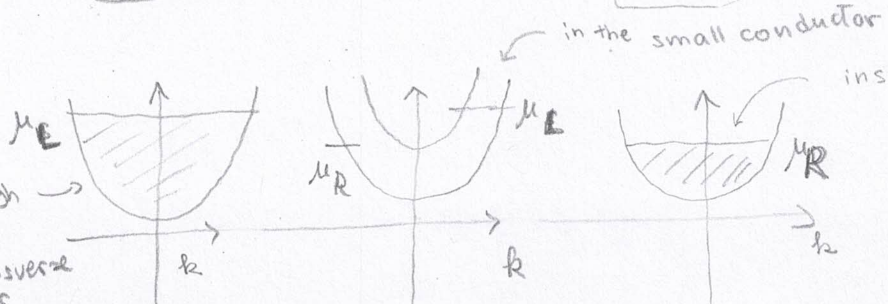
(note: $G_0 \approx 77.5\ \mu S$) (note: resistivity of Cooper at $T = 20°C$ $1.7\ 10^{-8}\ \Omega m$)

∘ The quantity $\left(\frac{h}{2e^2 M_{open}}\right) \equiv G_c^{-1}$ is called contact resistance

it arises from the interface between the ballistic conductor and the reservoirs; it reflects the fact that the reservoirs have ∞ many transverse modes, while the ballistic wire has only few of them ⇒ redistribution of the current among the current carrying states at the interface yields the resistance

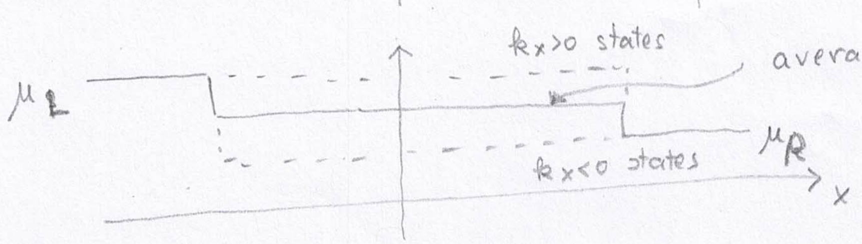


the Fermi levels for k_x and $-k_x$ are different in the small conductor



very high density of transverse modes

inside the reservoirs both k and $-k$ have nearly the same Fermi level



average value of the electroch. potential is flat inside the conductor
⇓
voltage drop is at the interface
⇓
resistance at interface

1.6 Ohm's law (stripe with $L \gg \ell_{el} > \ell_\varphi$)

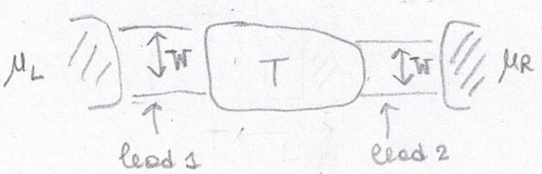
For a wide conductor with many transverse modes, and with $L \gg \ell_{el}$, the Ohm's law $G^{-1} \sim \frac{L}{W\sigma}$ should be recovered.

i) Consider a wide conductor with impurities .

View it as a generic conductor with transmission probability $T(E_F)$. Then Eq. (1.21) still holds $\Rightarrow$

$$G = \frac{2e^2}{h} M T \cong \frac{2e^2}{h} W \frac{k_F}{2\pi} T \qquad (1.22)$$

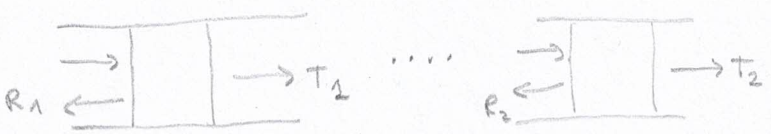
$$M \sim \frac{k_F}{\Delta k} = \frac{k_F}{2\pi/W}$$



ii) Let us now proof that the average transmission probability through a conductor of length $L \gg \ell_{el}$ is given by

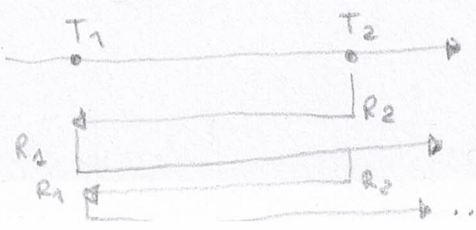
$$T = \frac{L_0}{L + L_0} \qquad (1.23)$$

with L_0 a characteristic length $\sim L_{el}$. To this extent we have to calculate the transmission probability through a chain of scatterers. Consider just two of them



$\Rightarrow T_{12}$?

Because two resistors in series have also reflection probabilities (neglecting all phase information $\rightarrow$ incoherent scattering) the transmission probability T_{12} must account for all multiple reflections



$T_1 T_2$

$T_1 R_2 R_1 T_2$

$\ldots \quad T_1 (R_2 R_1)^2 T_2 \quad \ldots$

independent scattering

Summing over all processes

$$\Rightarrow \quad T_{12} = \frac{T_1 T_2}{1 - R_1 R_2}$$

using $\quad T_1 = 1 - R_1, \quad T_2 = 1 - R_2$

$$\Rightarrow \quad \frac{1 - T_{12}}{T_{12}} = \frac{1 - T_1}{T_1} + \frac{1 - T_2}{T_2}$$ additive property of $\frac{1-T_i}{T_i}$!!!

$\Rightarrow$ N scatterers in series

$$\frac{1 - T(N)}{T(N)} = \sum_{i=1}^{N} \frac{(1 - T_i)}{T_i} = N \, \frac{1-T}{T} \quad \Rightarrow \quad T(N) = \frac{T}{N(1-T) + T}$$

$\uparrow$ identical scatterers

But the nr. of scatterers in a conductor of length L is $N \sim \frac{L}{\ell_{ee}}$

$$\Rightarrow \quad T(N) \cong \frac{\ell_{ee} T}{L(1-T) + \ell_{ee} T} = \frac{\ell_{ee} T/(1-T)}{L + \underbrace{\ell_{ee} T/(1-T)}_{L_0}} = \frac{L_0}{L + L_0}$$

iii)

$$(1.22) + (1.23) \Rightarrow \quad G = \frac{W}{L + L_0} \, L_0 \left(e^2 \frac{k_F}{\pi h} \right) = \frac{\sigma W}{L + L_0}, \quad \sigma = \frac{n_{2D} e^2 \tau}{m}$$

if $L_0 \sim v_F \tau$

$k_F = \sqrt{2\pi n}$

$m v_F = \hbar k_F$

or $$\boxed{G^{-1} = \frac{L + L_0}{\sigma W} = \underset{\uparrow \text{ohm}}{G_s^{-1}} + G_c^{-1}} \quad (1.24)$$

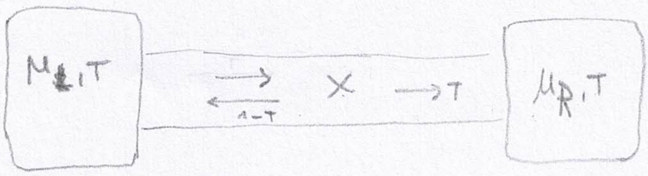
(note; 3D $G^{-1}_{\text{ohm}} = \frac{L}{\sigma W \cdot d}$, $\sigma = \frac{n_{3D}}{m}$

is the series combination of a contact resistance and of an "actual" resistance. Alternatively, we can also write (1.22) as

$$\boxed{G^{-1} = \frac{h}{2e^2} \frac{1}{MT} = \frac{h}{2e^2 M} + \frac{h}{2e^2 M} \frac{1-T}{T} = G_c^{-1} + G_s^{-1}} \quad (1.25)$$

resistance due to scatterers

Note: potential drop due to a scatterer



$\times$ = scatterer e.g. QPC

In this situation the distribution function in the wire is far from equilibrium. Let us denote by $f^{\gtrless}$ the distribution for right / left movers. We have two easy cases

left of scatterer $f^>(E) = \theta(\mu_L - E)$

right of scatterer $f^<(E) = \theta(\mu_R - E)$

and two more difficult ones for the left/right movers after the scatterer

near left $f^<(E) \cong T\theta(\mu_R - E) + (1-T)\,\theta(\mu_L - E)$

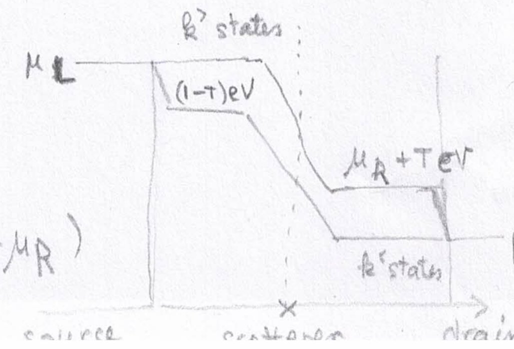
near right $f^>(E) \sim T\,\theta(\mu_L - E) + (1-T)\,\theta(\mu_R - E)$

and a bit more away from the scatterer such that energy relaxation has occurred

far left $f^<(E) \sim \theta(F^< - E)$

far right $f^>(E) \sim \theta(F^> - E)$

Imposing that that the total number of electrons integrated over E stays the same, yields the non equilibrium electrochemical potentials $F^<$ and $F^>$:



$$F^< = \mu_R + (1-T)(\mu_L - \mu_R)$$

$$F^> = \mu_R + T(\mu_L - \mu_R) = \mu_L - (1-T)(\mu_L - \mu_R)$$
$$= \mu_R + T(\mu_L - \mu_R)$$

$\Rightarrow$ We conclude that :

$$eV_S \equiv (1-T)(\mu_S - \mu_D) \qquad \text{potential drop at the scatterer}$$

$$eV_C \equiv T\,(\mu_S - \mu_D) \qquad \text{potential drop at interface with reservoirs}$$

<u>Note</u>: $T(\mu_S - \mu_D)$ is indeed the potential we would expect for a current $I = \frac{2e}{h} MT\,[\mu_S - \mu_D]$

flowing through the scatterer resistance

$$G_S^{-1} = \frac{h}{2e^2 M}\,\frac{1-T}{T}$$ in series with the contact resistance

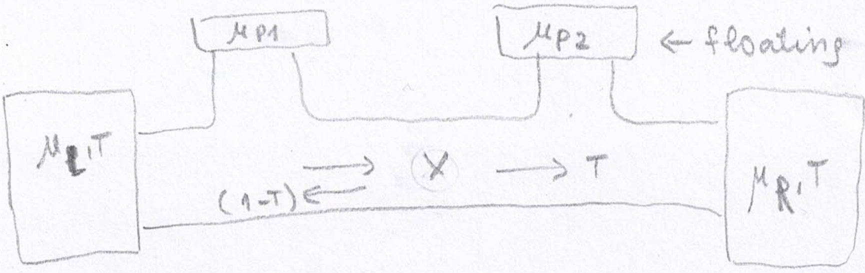
$$G_C^{-1} = \frac{h}{2e^2 M}$$

$$\Rightarrow G^{-1} = G_S^{-1} + G_C^{-1} = \frac{h}{2e^2 M}\left(\frac{1-T}{T} + 1\right) = \frac{h}{2e^2 M}\,\frac{1}{T}$$

For further discussion Ch. 2.3 of S. Datta

(Later:)

<u>Note</u>: To probe the voltage drop across a scatterer, a four probe measurement is usually performed



$\leftarrow$ floating terminals

Hall bridge

voltage probes located before and after the scatterer

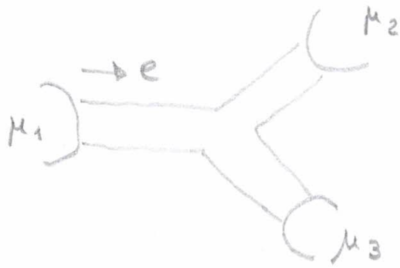
$$\Rightarrow \text{we expect } \mu_{P1} - \mu_{P2} = (1-T)(\mu_L - \mu_R) \Rightarrow R_{4t} = \frac{(\mu_{P1} - \mu_{P2})/e}{I} = \frac{h}{2e^2 M}\,\frac{(1-T)}{T}$$

1.7 Outlook

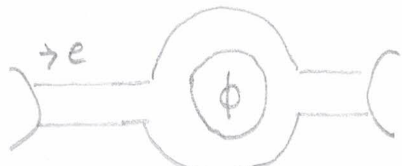
In the following we shall investigate the impact of size quantization, quantum coherence, quantum statistics and many-body interactions on the transport properties of complex meso/nano devices.

E.g.

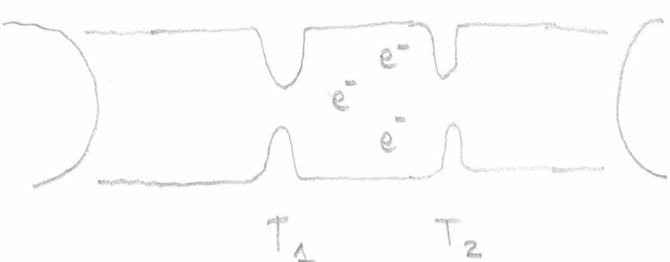
a) Beam splitter



b) Aharonov-Bohm interferometer



c) Fabry-Perot interferometer / quantum dot



$E_C = \frac{e^2}{C}$ charging energy

d) Diffusive, phase coherent conductor

$l_e < L < l_\phi$ impurity scattering

Needed: Transport formalism(s) coping with complex geometries, many-body aspects, disorder