4.2 Landauer formula and the conductance of a quantum point contact

Let us now consider a generic mesoscopic conductor with $L \to \infty$

I.e. in the same spirit as before, the reservoirs only act as sources of electrons and set μ_0, T in the mesoscopic system.

Such system can have complicated geometries e.g. branches, like a ring or one or more constrictions acting as tunneling barriers. For such systems, the conductance is given, in the case of $\hat{V}=0$ (non interacting electrons) by eq. (4.22)

$$\lim_{\omega \to 0} \tilde{G}(\omega) = G = 2\pi\hbar \sum_{\lambda\lambda'} | i_{\lambda\lambda'}(x') |^2 \left(-\frac{\partial f(E_\lambda)}{\partial E_\lambda} \right) \delta(E_\lambda - E_{\lambda'}) \tag{4.22}$$

In the following, we apply this formula exemplarily to the case of a quantum point contact (QPC)

As we shall see, if we view the associated tunneling problem as a scattering problem, Eq. (4.22) can be recast in the form (at $T=0$)

$$G = \frac{2e^2}{h} \operatorname{Tr}\{t^\dagger t\} = \frac{2e^2}{h} \sum_n \Upsilon_n(\mu)$$

with t a transmission matrix and $\{\Upsilon_n\}$ the eigenvalues of $t^\dagger t$.

Because of the invariance of the trace it also holds

$$\mathrm{Tr}\{t^\dagger t\} = \sum_m (t^\dagger t)_m = \sum_m T_m$$

here $(t^\dagger t)_m \hat{=} T_m$ are transmission probabilities.

In general, $T_m \neq \tilde{T}_m$. They coincide, however, in the case in which reflection probabilities at the barrier are negligible.

<u>Note</u>: According to the continuity equation, the current matrix element $i_{\lambda\lambda'}(x)$ cannot depend on x in the steady state

From $\partial_x \hat{i}(x) + \partial_t \hat{\rho} = 0$,

It follows

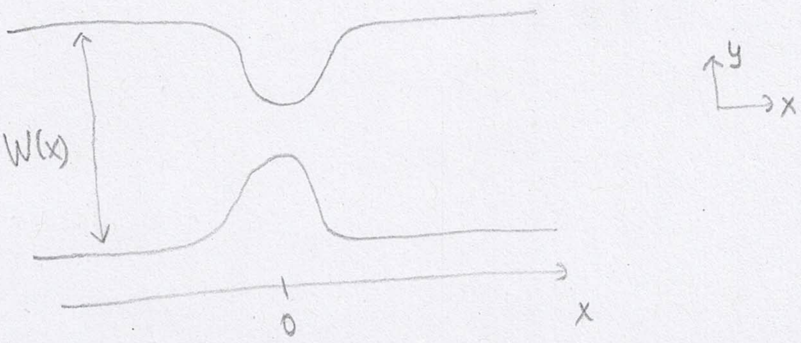
$$\langle \lambda | \partial_x \hat{i}(x) | \lambda' \rangle = -\langle \lambda | \dot{\hat{\rho}} | \lambda' \rangle = -\frac{i}{\hbar} \langle \lambda | [\hat{h}, \hat{\rho}] | \lambda' \rangle$$

$$= -\frac{i}{\hbar} (E_\lambda - E_{\lambda'}) \langle \lambda | \hat{\rho} | \lambda' \rangle = 0 \qquad \text{when } E_\lambda = E_{\lambda'}$$

which is ensured by the Dirac-delta in (4.22)

↳ $i_{\lambda\lambda'}(x)$ <u>can be calculated at a convenient position</u>

Conductance of a QPC



- Noninteracting electrons $\hat{V} = 0 \Rightarrow$ Eq. (4.33) applies

- Main task is to define the basis set $\{|\lambda\rangle\}$

I.e. one has to look for the eigenstates of $\hat{h} = \frac{\hat{p}^2}{2m} + U(\hat{x},\hat{y})$

which are solutions of the Schrödinger eq.

$$\hat{h}\,\psi(x,y) = \left[-\frac{\hbar^2}{2m}\left(\partial_x^2 + \partial_y^2\right) + U(x,y)\right]\psi(x,y) = E\,\psi(x,y)$$

The constriction breaks translational invariance along x

$\hookrightarrow$ k_x no longer good quantum number, E still is

- For smooth confinement, it is convenient to expand $\psi(x,y)$ in terms of the transverse eigenstates $\phi_{mx}(y)$, which however depend on position x:

$$\psi(x,y) = \sum_m \chi_m(x)\,\phi_{mx}(y) \qquad (4.23)$$

This is possible since, at any x, the $\phi_{mx}(y)$ solve the transverse Sch. eq

$$\left[-\frac{\hbar^2}{2m}\partial_y^2 + U(x,y)\right]\phi_{mx}(y) = \mathcal{E}_m(x)\,\phi_{mx}(y) \qquad (4.24)$$

Inserting (4.24) in (4.22) and integrating over y (cf. Exercise sheet 3)

$$\Rightarrow \boxed{\left[-\frac{\hbar^2}{2m}\partial_x^2 + \varepsilon_n(x)\right]\chi_n(x) = E\,\chi_n(x) + \delta_n(\{\chi_{n'}\})} \quad (4.25)$$

with

$$\delta_n = \frac{\hbar^2}{m}\sum_{n'}\int dy\, \phi_n^*(y)\left[(\partial_x \chi_{n'}(x))(\partial_x \phi_{n'x}(y)) + \frac{1}{2}\chi_{n'}(x)\,\partial_x^2\phi_{n'x}(y)\right]$$

- <u>consequences of confinement</u>

  i) Coupled equations for $\chi_n(x)$ due to the dependence of $\phi_{nx}(y)$ on x

  ii) momentum k_x no longer good quantum number ($\Rightarrow$ use E)

- <u>simplification 1: adiabatic approximation</u>

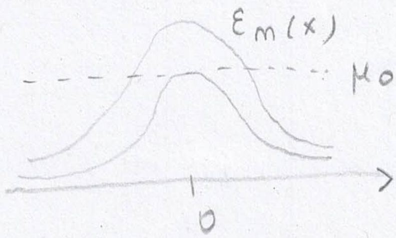
$$\partial_x \phi_{n'x}(y) \approx 0 \;\Leftrightarrow\; \delta_n \approx 0 \quad (4.26)$$

smooth variation of transverse mode along longitudinal direction

$\Rightarrow$ within adiabatic approximation $\chi_n(x)$ solves the 1D Schrödinger equation with <u>effective potential</u> $\varepsilon_n(x)$

<u>example</u>: hard wall confinement $\quad \phi_n\left(x, \pm\frac{W(x)}{2}\right) = 0 \;\forall n$

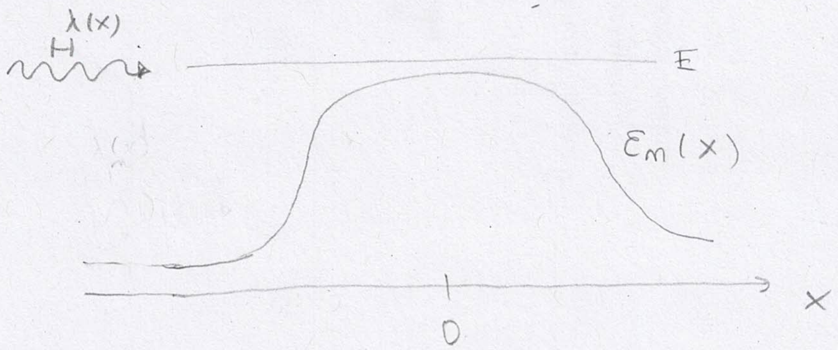
$$\hookrightarrow \begin{cases} \varepsilon_n(x) = \dfrac{\pi^2 n^2}{2m\, W^2(x)} \\[2ex] \phi_{nx}(y) = \sqrt{\dfrac{2}{W(x)}}\,\sin\left[n\pi\left(\dfrac{y}{W(x)} + \dfrac{1}{2}\right)\right] \end{cases}$$



$$\hookrightarrow \left[-\frac{\hbar^2}{2m}\partial_x^2 + \varepsilon_n(x)\right]\chi_n(x) = E\,\chi_n(x) \quad (4.27)$$

eq. for χ_n in adiabatic approximation

Smooth barrier : WKB approximation



smooth barrier : variation of $\varepsilon_m(x)$ small compared to $\lambda(x)$,

where $\lambda_m(x) = \frac{\hbar}{p_m(x)}$, $p_m(x) = \sqrt{2m(E - \varepsilon_m(x))}$ (4.27)

<u>WKB condition</u> $\boxed{\hbar |p'(x)| / p^2(x) \ll 1}$ (4.28)

i) "<u>classical" region</u> $E > \varepsilon_m(x=0)$

$$\chi_m(x) \approx \tilde{\chi}^+_m(x) = \frac{C_+}{\sqrt{p(x)}} e^{\frac{i}{\hbar}\int_{-\infty}^{x} dx'\, p_m(x')} \quad {}^{(*)} \qquad (4.29)$$

Solves Eq. (4.26) $\Rightarrow \tilde{\chi}_m = \tilde{\chi}_{mk(E)}$, $k = p/\hbar$

Further, it exists the "backward" solution $\tilde{\chi}^-_m(x) = \frac{C_-}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}\int_{-\infty}^{x} dx'\, p_m(x')}$

ii) tunneling region $E < \varepsilon_m(x=0)$

$$\tilde{\chi}_m(x) \approx \frac{D_\pm}{\sqrt{|p(x)|}} \exp \pm \int dx' |p(x')|$$

<u>Note</u>:

$$\partial_x \tilde{\chi}^+_m(x) = \frac{C_+}{\sqrt{p(x)}} \frac{i}{\hbar} p(x) \exp\left(\frac{i}{\hbar}\int_{-\infty}^{x} dx'\, p_m(x')\right) + C_+\, p(x)^{-3/2}\, p'(x)\, e^{\frac{i}{\hbar}\int_{-\infty}^{x} dx'\, p_m(x')}$$

(*) <u>Note</u>: Also "<u>standard" "plane wave" solution</u> is possible

$$\tilde{\chi}_m(x) = \frac{C_1}{\sqrt{L}} e^{i/\hbar \int_{-\infty}^{x} dx'\, p_m(x')} \qquad (4.29b)$$

$\Rightarrow$ use $\tilde{\chi}_m(x)$ or $\chi_m(x)$ depending on problem

Note: Relation between expectation values calculated using χ_n or $\tilde{\chi}_n$

Consider the case of a plane wave

$$\chi_{nk}(x) = \frac{1}{\sqrt{L}} e^{ik_n x} \quad , \quad \tilde{\chi}_{nk}(x) = \frac{C_+}{\sqrt{k_n}} e^{ik_n x}$$

For $\chi_n(x)$ it holds $|\chi_{nk}|^2 = 1$, i.e. the norm is conserved and independent of position, while for $\tilde{\chi}_n$ is

$$\tilde{\chi}^*_{nk} \hat{p}_x \tilde{\chi}_{nk} = \tilde{\chi}^*_{nk}(x) \left(\frac{-i}{\hbar} \partial_x \right) \tilde{\chi}_{nk}(x) = \frac{1}{\hbar} |C_+|^2$$

i.e., the particle's flux is independent of x.

This property is convenient for tunneling problems when matching conditions at the tunneling barrier are imposed and for which E and not k is a defined quantity.

We wish now to relate expectation values. Set $\begin{cases} \langle x | k \rangle = \frac{1}{\sqrt{L}} e^{ikx} \\ \langle x | \tilde{k} \rangle = \frac{1}{\sqrt{k}} e^{ikx} \end{cases}$

↳ for a generic operator $\hat{A}$

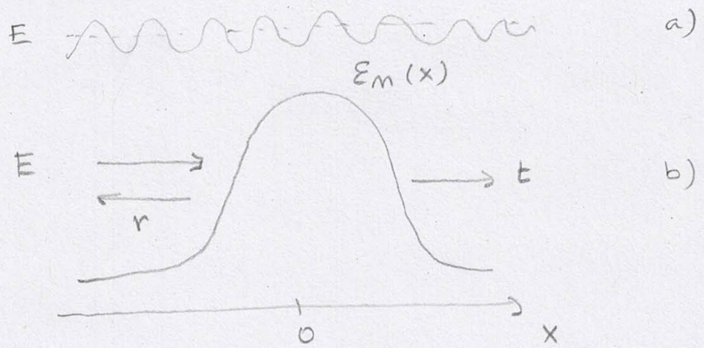
$$\sum_{k>0} \langle k | \hat{A} | k \rangle = \frac{L}{2\pi} \int_0^\infty dk \, \langle k | \hat{A} | k \rangle = \frac{1}{2\pi} \int_0^\infty dk \, k \, \langle \tilde{k} | \hat{A} | \tilde{k} \rangle$$

$$= \frac{1}{2\pi} \int_0^\infty dE \, \frac{k}{dE/dk} \langle \tilde{k} | \hat{A} | \tilde{k} \rangle$$

⇒ if $E = \varepsilon_n + \frac{\hbar^2 k^2}{2m}$ is $\frac{dE}{dk} = \frac{\hbar^2}{m} k$ ⇒ $\boxed{\sum_{k>0} \langle k | \hat{A} | k \rangle = \frac{m}{2\pi \hbar^2} \int_0^\infty dE \, \langle \tilde{k} | \hat{A} | \tilde{k} \rangle}$

(4.31)

- Consequences of barrier (WKB)



$$\begin{cases} E > \mathcal{E}_m(x=0) & \text{perfect transmission (WKB)} \\ E < \mathcal{E}_m(x=0) & \text{transmission and reflection} \end{cases}$$

simplification 2 : vanishing transmission if $E < \mathcal{E}_m(x=0)$

↳ only $\begin{cases} \text{open channels} & E > \mathcal{E}_m(x=0) \\ \text{closed} \quad '' & E < \mathcal{E}_m(x=0) \end{cases}$

i.e. the barrier is not a scatterer but only modulates phase and amplitude of $\tilde{\chi}_n(x)$ cf. (4.29) or $\chi_n(x)$ cf. (4.29b)

↳ appropriate basis : $\{|\lambda, \sigma\rangle\}$ with $|\lambda\rangle = |m, k(E)\rangle$, σ spin

where

$$\langle \vec{r} | m, k(E) \rangle = \chi_{m_k}(x) \, \phi_{mx}(y) = \psi_\lambda(\vec{r}), \quad \hat{h} \, |m \, k(E)\rangle = E \, |m \, k(E)\rangle$$

and $E = \mathcal{E}_m(x) + \dfrac{\hbar^2 k^2(x)}{2m}$

current operator $\hat{I}(x) = \sum_{\substack{\lambda\lambda' \\ \sigma}} i_{\lambda\lambda'}(x) \hat{c}^+_{\lambda\sigma} \hat{c}_{\lambda\sigma}$

- relation of field operators to $\hat{c}_\lambda$ operators

$$\hat{\psi}^+_\sigma(\vec{r})|0\rangle = |\vec{r},\sigma\rangle = \sum_{\lambda'\sigma'} |\lambda'\sigma'\rangle\langle\lambda'\sigma'|\vec{r}\sigma\rangle = \sum_{\lambda'\sigma'} \psi^*_{\lambda'}(\vec{r})|\lambda'\sigma'\rangle\,\delta_{\sigma'\sigma}$$

$$\hookrightarrow \quad \hat{\psi}^+_\sigma(\vec{r}) = \sum_\lambda \psi^*_\lambda(\vec{r})\,\hat{c}^+_{\lambda\sigma} \qquad (4.32)$$

- current operator in $\{|\lambda\sigma\rangle\}$ basis

$$\hat{I}(x) = \sum_{\lambda_1\lambda_2}\sum_\sigma \left(\frac{-ie\hbar}{2m}\right)\int dy\left[\psi^*_{\lambda_1}(\vec{r})\,\partial_x\psi_{\lambda_2}(\vec{r}) - \partial_x\psi^*_{\lambda_1}(\vec{r})\,\psi_{\lambda_2}(\vec{r})\right]$$

$$\cdot\ \hat{c}^+_{\lambda_1\sigma}\,\hat{c}_{\lambda_2\sigma}$$

$$\Rightarrow \boxed{i_{\lambda_1\lambda_2}(x) = \left(\frac{-ie\hbar}{2m}\right)\int dy\left[\psi^*_{\lambda_1}(\vec{r})\,\partial_x\psi_{\lambda_2}(\vec{r}) - \partial_x\psi^*_{\lambda_1}(\vec{r})\,\psi_{\lambda_2}(\vec{r})\right]} \qquad (4.33)$$

In general $i_{\lambda_1\lambda_2}(x)$ is <u>not diagonal</u> in <u>the channel index</u> m.

A decoupling of channels requires the adiabatic approximation:

$$\psi_\lambda(\vec{r}) = \langle\vec{r}|\lambda\rangle = \langle\vec{r}|m\underset{\uparrow\, k=k(E)}{k}\rangle = \chi_{mk}(x)\,\phi_{mx}(y)$$

$$\Rightarrow \int dy\,\psi^*_{\lambda_1}(\vec{r})\,\partial_x\psi_{\lambda_2}(\vec{r}) = \int dy\,\chi^*_{mk}(x)\,\phi^*_{mx}(y)\,\partial_x\left(\chi_{mk'}(x)\,\phi_{mx}(y)\right)$$

adiabatic approximation $\approx \chi^*_{mk}(x)\,\partial_x\chi_{mk'}(x)\underbrace{\int dy\,\phi^*_{mx}(y)\,\phi_{mx}(y)}_{\delta_{mm}}$

$\partial_x\phi_{mx}(y) \simeq 0$

$$\Rightarrow \boxed{\int dy\,\psi^*_{\lambda_1}(\vec{r})\,\partial_x\psi_{\lambda_2}(\vec{r}) = \chi^*_{mk}(x)\,\partial_x\chi_{mk'}(x)\,\delta_{mm}} \qquad (4.34)$$

in adiabatic approximation

Conductance of adiabatic and "opaque" QPC

opaque QPC $\Rightarrow$ no tunneling through barrier

$$\Rightarrow \begin{cases} \chi^{+}_{mk}(x) = \dfrac{1}{\sqrt{L}}\, e^{i\int_{x_0}^{x} dx'\, k_m(x')} & \text{forward propagating} \\[2em] \chi^{-}_{mk}(x) = \dfrac{1}{\sqrt{L}}\, e^{-i\int_{x_0}^{x} dx'\, k_m(x')} & \text{backward propagating} \end{cases}$$

with $k_m(x) = \sqrt{2m(E-\varepsilon_m(x))}\,/\hbar = k_m(E)$

$$\Rightarrow \partial_x \chi^{\pm}_{mk}(x) \underset{\text{WKB}}{\cong} \pm i\, k(x)\, \chi^{\pm}_{mk}(x)$$

by accounting for forward and backward solutions the same result as for the plane waves is obtained $\left.\begin{matrix}+k\\-k\end{matrix}\right\} \to k \gtrless 0$

$$\Rightarrow i_{\lambda\lambda'}(x) = \langle \lambda | \hat{i}(x) | \lambda' \rangle = -\frac{ie\hbar}{2m}\left[\chi^{*}_{mk}(x)\,\partial_x \chi_{mk'}(x) - \partial_x \chi^{*}_{mk}\, \chi_{mk'}(x)\right]\delta_{mm'}$$

$\lambda = m, k(E)$
$\lambda' = m, k'(E)$

$$\Rightarrow i_{\lambda\lambda'}(x) = \frac{e\hbar}{m}\,\frac{(k(x)+k'(x))}{2L}$$

with $k \gtrless 0$, $k' \lessgtr 0$ and $|k_m| = \sqrt{2m(E-\varepsilon_m(x=0))}\,/\hbar$

This result is the same as for the perfect wire $\Rightarrow$ Eq. (4.18b) for G. Since, in addition, G is independent of where x is calculated we can evaluate the conductance far away from the barrier

$$\Rightarrow G = 2\pi\hbar \left(\frac{e\hbar}{2mL}\right)^2 \sum_{qkm} (2k+q)\left(-\frac{\partial f}{\partial \varepsilon_k}\right)\delta(\varepsilon_k - \varepsilon_{k+q}) \underset{\text{cf. perfect wire}}{\cong} \frac{2e^2}{h}\sum_m f(\varepsilon_m(x=0))$$

Non adiabatic QPC with finite transmission

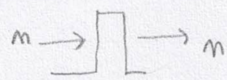
We release now both simplifications we made so far

i) adiabatic approximation no longer valid

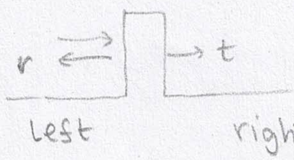
e.g. Steep barrier ⇒ $\partial_x \phi_n(y) \neq 0$.

⇒ mixing of channels through the barrier

$i_{mm'} \neq i_{mm'} \delta_{mm'}$



ii) barrier is not opaque ⇒ tunneling possible



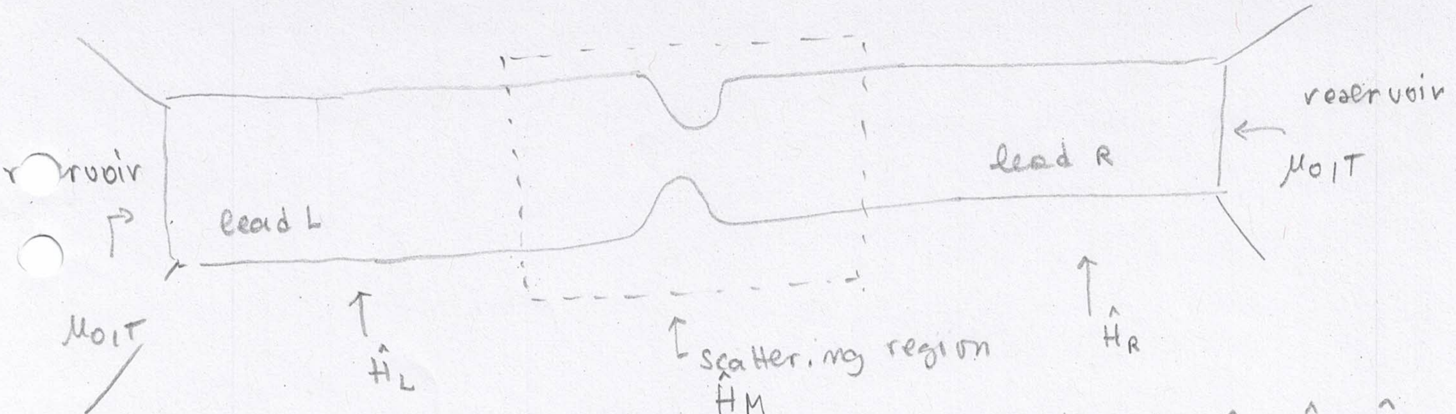
reflection at the barrier & tunneling must be included

⇒ traveling states $\psi^{(n)}_{\alpha m E}(\vec{r})$, $\alpha = L, R$

states of energy E with an incoming wave from left / right and that far away from the scatterer, can be described as linear combinations of forward propagating ($k>0$) and backward propagating ($k<0$) solutions $\left\{ \psi^{\pm}_{m\alpha k(E)}(\vec{r}) = \frac{1}{\sqrt{L}} \phi_{m\alpha}(y)\, e^{\pm ikx}, \quad \alpha = L, R \right\}$

of the Schrödinger eq. for perfect wires (leads) at the left / right ($\alpha = L, R$) of the scatterer

• Specifically; we use so called "scattering" boundary conditions, whereby we imagine that, if we are far away from the barrier (scatterer), we can assume to have scattering free leads (i.e. pieces of an ideal wire) connected to reservoirs. The latter μ_0, T for the whole system (global equilibrium)



$$\hat{H}_S = \hat{H}_L + \hat{H}_M + \hat{H}_R$$

scattering boundary conditions:

$$\begin{cases} \lim\limits_{x \to -\infty} \hat{H}_S = \hat{H}_L \\ \lim\limits_{x \to +\infty} \hat{H}_S = \hat{H}_R \\ \hat{H}_\alpha = \sum\limits_{mk} h_{\alpha\lambda} \hat{C}^{+}_{\alpha m k} C_{\alpha m k} \end{cases}$$

, isolated lead eigenfunctions of $\hat{h}_L, \hat{h}_R$ $\quad \psi_{mk}(\vec{r}) = \frac{1}{\sqrt{L}} \phi_{m\alpha}(y) e^{ikx}$

energies $\quad E_{m\alpha k} = \varepsilon_{m\alpha} + \frac{\hbar^2 k^2}{2m}$

Importantly, energy is conserved by an electron traveling from left $\to$ right or right $\to$ left

Traveling states

For an incoming wave from lead L

$$\Psi_{mLE}(x,\vec{r}_\perp) = \begin{cases} \psi^{+}_{mLk(E)}(\vec{r}) + \sum_{n=1}^{N_L} R_{nm}\, \psi^{-}_{nLk(E)}(\vec{r}) & (x,\vec{r}_\perp) \in L \\ \psi_{M,E}(\vec{r}) & (x,\vec{r}_\perp) \in M \\ \sum_{n=1}^{N_R} T_{nm}\, \psi^{+}_{nRk(E)}(\vec{r}) & (x,\vec{r}_\perp) \in R \end{cases} \quad (4.35)$$

where N_α nr. of transverse channel of lead α.

Further $\psi_{ME}(\vec{r})$ in the middle region are complicated functions which are in principle determined by imposing the continuity of $\Psi_{mLE}(x,\vec{r}_\perp)$ and its derivative at the boundaries.

Similarly, for an incoming wave from lead R

$$\Psi_{mRE}(x,\vec{r}_\perp) = \begin{cases} \sum_{n=1}^{N_L} T'_{nm}\, \psi^{-}_{nLk(E)}(\vec{r}) & (x,\vec{r}_\perp) \in L \\ \psi_{ME}(\vec{r}) & (x,\vec{r}_\perp) \in M \\ \sum_{n=1}^{N_R} R'_{nm}\, \psi^{+}_{nRk(E)}(\vec{r}) + \psi^{-}_{mRk(E)}(\vec{r}) & (x,\vec{r}_\perp) \in R \end{cases} \quad (4.36)$$

<u>note</u>: R_{nm} and T_{nm} (R'_{nm} and T'_{nm}) are <u>not</u> reflection and transmission amplitudes normalized to the flux

Current matrix

$$i_{\lambda\lambda'}(x) = \frac{e\hbar}{2im} \int dy \left[\psi^*_\lambda(\vec r) \frac{\partial \psi_{\lambda'}(\vec r)}{\partial x} - \frac{\partial \psi^*_\lambda(\vec r)}{\partial x} \psi_{\lambda'}(\vec r) \right]$$

with $\vec r = (x, y)$, $\lambda = (m \alpha E)$.

i) According to the continuity equation, the current matrix elements are independent of x and can be calculated in the L or R regions at our convenience.

ii) Further, from the conductance formula (4.22)

$$G = 2\pi\hbar \sum_{\lambda\lambda'} |i_{\lambda\lambda'}(x)|^2 \left(-\frac{\partial f}{\partial E_\lambda} \right) \delta(E_\lambda - E_{\lambda'}) \qquad (4.22)$$

2) energy is conserved : $E_\lambda = E_{\lambda'}$

We look for

$$i_{m\alpha k_m(E),\, m'\alpha' k_{m'}(E')}(x)$$ with $E = E'$

<u>note</u>: we use in general

$$k'_m(E') = \frac{\sqrt{2m(E' - \varepsilon_m)}}{\hbar} = k'_m$$

$$k_m(E) = \sqrt{2m(E - \varepsilon_m)}/\hbar = k_m$$

Final result (see Proof pages 32-36)

$\underline{x \in L}$

a) $$\boxed{i_{mLE\, m'LE}(x) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\left[\delta_{mm'} - (r^+ r)_{mm'}\right]} \quad (4.37\,a)$$

lecture 4.31a

with

$$\boxed{r_{mm} \equiv R_{mm}\sqrt{\frac{v_m}{v_m}}} \quad (4.38)$$ reflection coefficient at lead L

and $v_m = \frac{1}{\hbar}\frac{\partial E}{\partial k_m} = \frac{\hbar k_m}{m}$

b) $$\boxed{i_{mRE\, m'RE}(x) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\left[-(t'^+)t'\right]_{mm'}} \quad (4.37b)$$

with

$$\boxed{t'_{mm} \equiv J'_{mm}\sqrt{\frac{v_m}{v_m}}} \quad (4.39)$$ transmission coefficient at lead L

c) $$\boxed{i_{mLE\, m'RE}(x) = -\frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\,(r^+ t')} \quad (4.37c)$$

d) $$\boxed{i_{mRE\, m'LE}(x) = -\frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\,(t'^+ r)} \quad (4.37d)$$

Analogously, $\underline{x \in R}$ (Proof: page 37–39)

a) $$\boxed{i_{mLE\, m'LE}(x) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\,(t^{+}t)_{mm'}} \qquad (4.40a)$$

lecture 4.32a

with $$\boxed{t_{mm} = J_{mm}\sqrt{\frac{v_{m'}}{v_m}}} \qquad (4.41)$$ transmission coefficient at lead R

b) $$\boxed{i_{mRE\, m'RE}(x) = -\frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\left[\delta_{mm'} - (r'^{+}r')_{mm'}\right]} \qquad (4.40b)$$

with $$\boxed{r'_{mm} = R'_{mm}\sqrt{\frac{v_m}{v_n}}} \qquad (4.42)$$

c) $$\boxed{i_{mLE\, mRE}(x) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\cdot(t^{+}r') \qquad (4.40c)}$$

d) $$\boxed{i_{mRE\, mLE}(x) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}}\,(r'^{+}t) \qquad (4.40d)}$$

Relation between transmission and reflection coefficients

$i_{\lambda\lambda'}(x)$ independent of x

a) current $i_{mLE\,m'LE}(x)$ given by (4.37a) and (4.40a) is the same

$$t^{+}t = 1\!\!1 - r^{+}r \quad \Rightarrow \quad \boxed{t^{+}t + r^{+}r = 1\!\!1} \qquad (4.43a)$$

b) similarly for $i_{mRE\,m'RE}(x)$

$$t'^{+}t' = 1\!\!1 - r'^{+}r' \quad \Rightarrow \quad \boxed{t'^{+}t' + r'^{+}r' = 1\!\!1} \qquad (4.43b)$$

c) from expression for cross terms (4.38c) = (4.40c)

$$-r^{+}t' = t^{+}r' \quad \Rightarrow \quad \boxed{r^{+}t' + t^{+}r' = 0} \qquad (4.43c)$$

Scattering matrix

Interestingly, these relations reveal that r, r', t and t' are simply the elements of the scattering matrix

$$S = \begin{pmatrix} r(E) & t'(E) \\ t(E) & r'(E) \end{pmatrix} \qquad (4.44c)$$

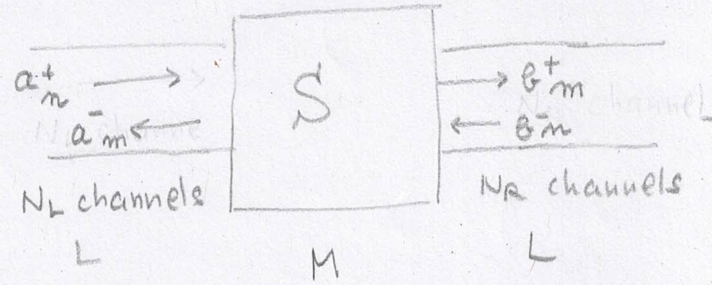
which relates the outgoing flux amplitudes $\sqrt{v_m}\,\psi^{-}_{mLR},\ \sqrt{v_m}\,\psi^{+}_{mRR}$ to incoming ones $\sqrt{v_m}\,\psi^{+}_{mLR},\ \sqrt{v_m}\,\psi^{-}_{mRR}$

note: $S = \left(\begin{array}{c|c} N_L \times N_R & N_L \times N_R \\ \hline N_R \times N_L & N_R \times N_R \end{array}\right)$

Define the vector $\begin{cases} \vec{C}^{in} & \text{of incoming } \underline{\text{flux}}\ \underline{\text{amplitudes}} \\ \vec{C}^{out} & \text{" outgoing "} \end{cases}$

$$\vec{C}^{in} = \begin{pmatrix} a_1^+ \\ a_2^+ \\ \vdots \\ a_{N_L}^+ \\ b_1^- \\ b_2^- \\ \vdots \\ b_{N_R}^- \end{pmatrix}, \qquad \vec{C}^{out} = \begin{pmatrix} a_1^- \\ a_2^- \\ \vdots \\ a_{N_L}^- \\ b_1^+ \\ \vdots \\ b_{N_R}^+ \end{pmatrix}$$

$$\rightarrow \boxed{\vec{C}^{out} = S\, \vec{C}^{in}} \quad (4.45)$$



E.g. for the traveling state (4.35)

$$\vec{C}^{in} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ a_n^+ = \sqrt{v_n}\cdot 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}}_{(N_L+N_R)\times 1}, \qquad \vec{C}^{out} = \begin{array}{cc} \begin{smallmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{smallmatrix} & \\ N_L \times N_L & N_L \times N_R \end{array} \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{v_n} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1^- \\ \vdots \\ a_{N_L}^- \\ b_1^+ \\ \vdots \\ b_{N_R}^+ \end{pmatrix}}_{(N_L+N_R)\times 1}$$

$N_R \times N_L \qquad N_R \times N_R$

$$(C^{out})_{m1} = \sum_j S_{mj} (C^{in})_{j1} = S_{mn} \qquad n \in N_L$$

$$\Rightarrow \vec{C}^{out} = \begin{pmatrix} r_{1n}\sqrt{v_n} \\ r_{2n}\sqrt{v_n} \\ \vdots \\ t_{1n}\sqrt{v_n} \\ \vdots \\ t_{N_R n}\sqrt{v_n} \end{pmatrix} = \begin{pmatrix} R_{1n}\sqrt{v_1} \\ R_{2n}\sqrt{v_2} \\ \vdots \\ J_{1n}\sqrt{v_1} \\ \vdots \\ J_{N_R n}\sqrt{v_{N_R}} \end{pmatrix}$$

in agreement with (4.35)

The relations (4.43a) – (4.43c) thus simply ensure flux conservation and hence the unitarity of the S-matrix,

$$S^\dagger S = 1 \qquad (4.46)$$

Proof:

$$S^\dagger S = \begin{pmatrix} r^\dagger & t^\dagger \\ t'^\dagger & r'^\dagger \end{pmatrix} \begin{array}{cc} N_L\times N_L & N_L\times N_R \\ \multicolumn{2}{c}{\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}} \\ N_R\times N_L & N_R\times N_R \end{array} = \begin{pmatrix} r^\dagger r + t^\dagger t & r^\dagger t' + t^\dagger r' \\ t'^\dagger r + r'^\dagger t & t'^\dagger t' + r'^\dagger r' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Further $\underline{S^\dagger S = 1}$ (4.45b) yields the additional relations

$$\boxed{\begin{aligned} 1 &= r' r'^\dagger + t t^\dagger = r r^\dagger + t' t'^\dagger \\ 0 &= r t^\dagger + t' r'^\dagger = t r^\dagger + r' t'^\dagger \end{aligned}} \qquad \begin{aligned} &(4.43d) \\ &(4.43e) \end{aligned}$$

For system with time-reversal symmetry, $\hat{H} = \hat{H}^*$, also implies

$$S = S^T \quad (4.46c)$$

i.e. the S-matrix is symmetric.

This property is important for the study e.g. of disordered systems.

From $\begin{cases} \mathbb{1} = t^\dagger t + r^\dagger r \\ \mathbb{1} = r r^\dagger + t' t'^\dagger \end{cases} \Rightarrow \boxed{\mathrm{Tr}\{t^\dagger t\} = \mathrm{Tr}\{t' t'^\dagger\} = \mathrm{Tr}\{t'^\dagger t'\}}$ (4.43f)

expression of $i_{\lambda\lambda'}(x)$ at $x \in R$

From previous relations (4.40a) – (4.40d) and using 4.43b, 4.43c

$$i_{m\alpha E\, m'\alpha' E}(x \in R) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}} \begin{pmatrix} (t^+t)_{mm'} & (t^+r')_{mm'} \\ (-t'^+r)_{mm'} & -(t'^+t')_{mm'} \end{pmatrix}$$

$$\Rightarrow \boxed{i_{m\alpha E\, m'\alpha' E}(x) = \frac{e}{L}\sqrt{v_m}\sqrt{v_{m'}} \; j_{\alpha m\, \alpha' m'}} \quad (4.47)$$

where $$j = \begin{pmatrix} t^+t & t^+r' \\ -t^+r & -t'^+t' \end{pmatrix} \quad (4.48)$$ is the scattering matrix

expression for G

According to (4.22)

$$G = 2\pi\hbar \sum_{\lambda\lambda'} |i_{\lambda\lambda'}|^2 \left(-\frac{\partial f}{\partial E_\lambda}\right) \delta(E_\lambda - E_{\lambda'})$$

$$= 2\pi\hbar \frac{e^2}{L^2} \sum_{mm'} \sum_{\substack{\alpha\alpha' \\ k_m k'_m}} v_m v_{m'} \, | j_{\alpha m\, \alpha' m'}^{(E)} |^2 \left(-\frac{\partial f}{\partial E}\right) \delta(E - E')$$

use $\frac{1}{L}\sum_{k_m(E)} v_m = \frac{1}{2\pi}\int dk_m v_m = \frac{1}{2\pi\hbar}\int dE$

$v = \frac{1}{\hbar}\frac{\partial E}{\partial k}$

$$\Rightarrow G = \frac{e^2}{h} \sum_{mm'} \sum_{\alpha\alpha'} \int dE \; | j_{\alpha m\, \alpha' m'}(E) |^2 \left(-\frac{\partial f}{\partial E}\right)$$

We need

$$\sum_{mm'}\sum_{\alpha\alpha'} |j^{(E)}_{\alpha m\, \alpha' m'}|^2 = \mathrm{Tr}\{ j^+ j\}$$

remember

$$\mathrm{Tr}\{A^+A\} = \sum_{n,m} \langle n|A^+|m\rangle\langle m|A|n\rangle = \sum_{nm} A^+_{nm} A_{mn}$$

$$= \sum_{nm} A^*_{mn} A_{mn} = \sum_{nm} |A_{mn}|^2$$

From (4.47)

$$\mathrm{Tr}\{j^+j\} = \mathrm{Tr}\{(t^+t)^2 + (t'^+t')^2 + r'^+ t t^+ r' + r^+ t' t'^+ r\} \overset{(*)}{=} 2\,\mathrm{Tr}\{t^+t\},$$

where the last equality can be demonstrated as follows

$$(*) \begin{cases} 1 = r r^+ + t' t'^+ \\ 1 = r' r'^+ + t t^+ \end{cases}$$

Hence

$$\mathrm{Tr}\{j^+j\} = \mathrm{Tr}\{ t^+(1 - r' r'^+) t + t'^+(1 - r r^+) t' + r'^+ t t^+ r' + r^+ t' t'^+ r\}$$

$$= \mathrm{Tr}\{(t^+t) + (t'^+t') - t^+ r' r'^+ t - t'^+ r r^+ t' + r'^+ t t^+ r' + r^+ t' t'^+ r\}$$

$$= 2\,\mathrm{Tr}\{t^+t\} - \mathrm{Tr}\{\underbrace{(r'^+t)^+ r'^+ t}_{A^+A}\} - \mathrm{Tr}\{\underbrace{(r^+t')^+ (r^+t')}_{B^+B}\} + \mathrm{Tr}\{AA^+\} + \mathrm{Tr}\{BB^+\}$$

$$= 2\,\mathrm{Tr}\{t^+t\}$$

$\Rightarrow$ It follows

$$\boxed{G = \frac{2e^2}{h} \int dE\, \mathrm{Tr}\{t^+t\}\left(-\frac{\partial f}{\partial E}\right)} \qquad (4.49)$$

Fazit: conductance as transmission

We have demonstrated that for any (non interacting) mesoscopic set-up which can be decomposed in left - middle - right part, with left/right perfect leads, the conductance reads

$$\boxed{G = \frac{2e^2}{h} \int dE \; \mathrm{Tr}\{t^\dagger t\} \left(-\frac{\partial f}{\partial E}\right)} \qquad (4.50)$$

This thus applies to the QPC but also to more complex systems.

Depending on the number of channels N_R in the right lead the resulting matrix $t^\dagger t$ is $N_R \times N_R$ and has N_R eigenvalues $\{T_n\}$

$$\hookrightarrow \mathrm{Tr}\{t^\dagger t\} = \sum_{i=1}^{N_R} T_n(E)$$

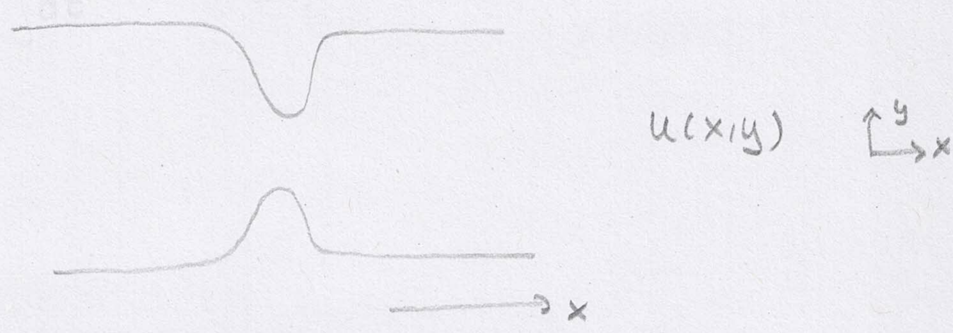
The whole problem thus reduces to the eigenvalues of the transmission matrix $T = t^\dagger t$.

The result

$$\boxed{G = \frac{2e^2}{h} \sum_n \int dE \; T_n(E) \left(-\frac{\partial f}{\partial E}\right)} \qquad (4.51)$$

is another form of the Landauer formula for the conductance

- Conductance of a QPC within saddle-point model



Consider a saddle point expansion of the confinement potential near its maximum (Büttiker 1990)

$$u(x,y) \sim \frac{1}{2} m_y \omega_y^2 y^2 - \frac{1}{2} m \omega_x^2 x^2 + V_0$$

$$\hookrightarrow T_m(E) = \frac{1}{\exp\left[\pi\left(E - V_0 - \left(m + \frac{1}{2}\right)\hbar\omega_x\right)/\hbar\omega_y\right] + 1}$$

Here we demonstrate the relations (4.37 a) – (4.37 d)

Proof:

$$i^{(x)}_{\lambda\lambda'} = \frac{e\hbar}{2im} \int dy \left[\psi^*_{\lambda}(\vec r) \frac{\partial \psi_{\lambda'}(\vec r)}{\partial x} - \frac{\partial \psi^*_{\lambda}(\vec r)}{\partial x} \psi_{\lambda'}(\vec r) \right]$$

the index $\lambda = m\alpha E_\lambda$, $\lambda' = m'\alpha' E_{\lambda'}$

depending on whether $\alpha = L, R$ and $\alpha' = L, R$ we find if $x \in L$

$\alpha = L,\ \alpha' = L$

$$i^{(x)}_{mLE_\lambda, m'LE_{\lambda'}} = \frac{e\hbar}{2im} \int dy \Big[\psi^{+*}_{mLE_\lambda} \partial_x \psi^+_{m'LE_{\lambda'}} - \partial_x \psi^+_{mLE_\lambda} \psi^+_{m'LE_{\lambda'}}(x)$$

$$+ \sum_{m'=1}^{N_L} R^*_{mm} \left(\psi^-_{m'LE_\lambda}(\vec r) \right)^* \partial_x \psi^+_{m'LE_{\lambda'}} - \sum_{m=1}^{N_L} R^*_{mm} \left(\partial_x \psi^{-*}_{m'LE_\lambda} \right) \psi^+_{m'LE_{\lambda'}}(x)$$

$$+ \sum_{m'=1}^{N_L} \sum_{m''=1}^{N_L} R^*_{mm} R_{m'm'} \left[\left(\psi^-_{mLE_\lambda}(\vec r) \right)^* \left(\partial_x \psi^-_{m'LE_{\lambda'}}(\vec r) \right) - \left(\partial_x \psi^-_{m'LE_\lambda} \right)^* \psi^-_{m'L} \right]$$

$$+ \sum_{m'=1}^{N_L} R_{m'm'} \left[\left(\psi^+_{mLE_\lambda} \right)^* \partial_x \psi^-_{m'LE_{\lambda'}} - \left(\partial_x \psi^{+*}_{mLE_\lambda} \right) \psi^-_{m'LE_{\lambda'}} \right]$$

since each transverse channel is orthogonal

$$\int dy\ \phi^*_n(y)\, \phi_m(y) = \delta_{nm}$$

From (*)

$$i^{(x)}_{mLE_\lambda\, m'LE_{\lambda'}} = \frac{e\hbar}{2miL}\Big[i\big(k'_{m}(E_{\lambda'}) + k_m(E)\big)\, e^{-i(k_m - k'_{m'})x}\,\delta_{mm'}$$

$$+ R^{*}_{m'm}\, e^{+i(k_{m'} + k'_{m'})x}\,\big(i\,(k'_{m'} - k_{m'})\big)$$

$$+ R_{mm'}\, e^{-i(k_m + k'_{m'})x}\,\big(-i\,(k'_m - k_m)\big)$$

$$+ \sum_{m=1}^{N_L} R^{*}_{mm}\, R_{mm'}\, e^{i(k_m - k'_m)x}\,\big(-i\,(k'_m + k_m)\big)\Big]$$

accounting now for the condition of energy conservation in G

$$E_\lambda = E_{\lambda'} \Rightarrow k_m(E) = k'_m(E)$$

and hence the two intermediate terms vanish

$$\boxed{\begin{aligned} i^{(x)}_{mLE\, m'LE} &= \frac{e\hbar k_m(E)}{2mL}\,\delta_{mm'}\cdot 2 - \sum_{m=1}^{N_L} R^{*}_{mm} R_{mm'}\, 2\cdot\frac{e\hbar}{2m}\, k_m(E) \\ &= \frac{e\hbar\sqrt{k_m}\sqrt{k_{m'}}}{2mL}\Big[\delta_{mm'} - \sum_{m=1}^{N_L} R^{*}_{mm} R_{mm'}\,\frac{k_m}{\sqrt{k_m}\sqrt{k_{m'}}}\Big] \end{aligned}}$$

↳ Eq. (4.37a) follows from (4.38)
↳ using $(AB)_{ij} = \sum_k A_{ik} B_{kj}$, $(A^{+}B)_{ij} = \sum_k A^{*}_{ki} B_{kj}$

(*)

$$\begin{cases} \psi^{\pm}_{mLE_\lambda} = \frac{1}{\sqrt{L}}\, e^{\pm i k_m x}\,\phi_m(y) , & \psi^{\pm}_{m'LE_{\lambda'}} = \frac{1}{\sqrt{L}}\, e^{\pm i k'_{m'} x} \\ k_m = \sqrt{2m(E_\lambda - \varepsilon_m)}/\hbar & k'_{m'} = \sqrt{2m(E_{\lambda'} - \varepsilon_{m'})}/\hbar \end{cases}$$

Similarly, for $x \in L$ and

$\alpha = R\,,\ \alpha' = R$ $\qquad E_\lambda = E_{\lambda'}$

$$\boxed{\begin{aligned} i^{(x)}_{nRE_\lambda\, n'RE_\lambda} &= -\frac{e\hbar}{2mL} \sum_{m=1}^{N_L} \left(J'_{mn}\right)^* J'_{mn'}\, k_m(E) \\ &= -\frac{e\hbar}{2mL} \sqrt{k_n}\sqrt{k_{n'}} \sum_{m=1}^{N_L} \left(t'_{mn}\right)^* t'_{mn'} \qquad x \in L \end{aligned}}$$

with

$$t'_{mn} \equiv J'_{mn} \sqrt{\frac{v_m}{v_n}}$$ the transmission coefficient at lead L

Eq. (4.37b) follows

Cross-terms $\alpha = L,\ \alpha' = R,\ x \in L$

$$i^{(x)}_{mLE\, m'RE} = \frac{e\hbar}{2im} \Big\{ \int dy \left[\psi^{+*}_{mLE} + \sum_{m=1}^{N_L} R^*_{mm} \psi^{-*}_{mLE} \right] \left[\sum_{m'=1}^{N_L} J'_{m'm'} \partial_x \psi^-_{m'LE} \right]$$

$$- \int dy\, \partial_x \left[\psi^+_{mLE} + \sum_{m=1}^{N_L} R_{mm} \psi^-_{mLE} \right]^* \sum_{m'=1}^{N_L} J'_{m'm'} \psi^-_{m'LE} \Big\}$$

$$= \frac{e\hbar}{2imL} \Big[e^{-ik_m x} J'_{mm'} e^{-ik_{m'} x} (-ik_{m'})$$

$$+ \sum_{m=1}^{N_L} R^*_{mm} J'_{mm'} e^{+ik_m x} e^{-ik_m x} (-ik_m)$$

$$- e^{-ik_m x} (-ik_m) J'_{mm'} e^{-ik_{m'} x}$$

$$- \sum_{m=1}^{N_L} R^*_{mm} J'_{mm'} e^{ik_m x} (ik_m) e^{ik_m x} \Big]$$

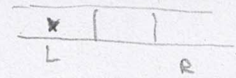
$$= \frac{e\hbar}{2imL} \Big[e^{-i(k_m + k_{m'})x} (-ik_{m'}) J'_{mm'} + \sum_{m=1}^{N_L} R^*_{mm} J'_{mm'} (-ik_m)$$

$$+ e^{-i(k_m + k_{m'})x} (+ik_m) J'_{mm'} - \sum_{m=1}^{N_L} R^*_{mm} J'_{mm'} ik_m \Big]$$

$$= \frac{e\hbar}{2miL} \Big[e^{-i(k_m + k_{m'})x} J'_{mm'} (-ik_{m'} + ik_m) - 2i \sqrt{k_m} \sqrt{k_{m'}} \underbrace{\sum_{m=1}^{N_L} r^*_{mm} t'_{mm'}}_{(r^+ t')_{mm'}}$$

$$= \frac{e\hbar}{2mL} \Big[e^{-i(k_m + k_{m'})x} J'_{mm'} \underbrace{(-k_{m'} + k_m)}_{=0} - 2\sqrt{k_m}\sqrt{k_{m'}}\, (r^+ t')_{mm'} \Big]$$

cross-terms $\alpha = R,\ \alpha' = L$ $\quad x \in L$



$$i_{mRE\,m'LE}(x) = \frac{e\hbar}{2mi}\Big\{ \int dy \Big[\sum_{m=1}^{N_L} J'_{mm}\, \psi^-_{mLE} \Big]^* \partial_x \Big[\psi^+_{m'LE} + \sum_{m'=1}^{N_L} R_{m'm'}\, \psi^-_{m'LE} \Big]$$

$$- \int dy\ \partial_x \Big[\sum_{m=1}^{N_L} J'_{mm}\, \psi^-_{mLE} \Big]^* \Big[\psi^+_{m'LE} + \sum_{m'=1}^{N_L} R_{m'm'}\, \psi^-_{m'LE} \Big]\Big\}$$

$$= \frac{e\hbar}{L2mi}\Big\{ e^{ik_{m'}x} (J'_{m'm})^* e^{ik_{m'}x} (ik_{m'})$$

$$+ \sum_{m=1}^{N_L} e^{+ik_m x} (J'_{mm})^* R_{mm'}\, e^{-ik_m x} (-ik_m)$$

$$- (J'_{m'm})^* e^{ik_{m'}x} (ik_{m'})\, e^{ik_{m'}x}$$

$$- \sum_{m=1}^{N_L} (J'_{mm})^* e^{ik_m x} (ik_m)\, R_{mm'}\, e^{-ik_m x} \Big\}$$

$$= \frac{e\hbar}{2mL}\Big\{ e^{i(k_{m'}+k_{m'})x} (J'_{m'm})^* (k_{m'} - k_{m'})$$

$$- \sum_{m=1}^{N_L} (J'_{mm})^* R_{mm'}\, 2k_m \Big\}$$

$$= \frac{e\hbar}{2mL}\Big\{ e^{i(k_{m'}+k_{m'})x} (J'_{m'm})^* \underbrace{(k_{m'} - k_{m'})}_{=0} - \underline{2\sqrt{k_m}\sqrt{k_{m'}}\ (t'^{\dagger} r)_{mm'}} \Big\}$$

cross-terms $\alpha = R$, $\alpha' = L$ $x \in R$

$$i^{(x)}_{mRE\, m'LE} = \frac{e\hbar}{2im}\left\{ \int dy \left[\psi^-_{mRk_m} + \sum_{m=1}^{N_R} R'_{mm} \psi^+_{mRk} \right]^* \partial_x \sum_{m'=1}^{N_R} \psi^+_{m'Rk_{m'}} J_{m'm'} \right.$$

$$\left. - \int dy\, \partial_x \left[\psi^-_{mRk_m} + \sum_{m=1}^{N_R} R'_{mm} \psi^+_{mRk} \right]^* \sum_{m'=1}^{N_R} \psi^+_{m'Rk_{m'}} J_{m'm'} \right\}$$

$$= \frac{e\hbar}{2imL}\left\{ e^{+ik_m x} e^{ik_m} (ik_m) J_{mm'} + \sum_{m=1}^{N_R} e^{-ik_m x} (R'_{mm})^* e^{ik_m x} (ik_m) J_{mm'} \right.$$

$$\left. - e^{ik_m x} (ik_m) e^{ik_m x} J_{mm'} - \sum_{m=1}^{N_R} (R'_{mm})^* e^{-ik_m x} (-ik_m) e^{ik_m x} J_{mm'} \right\}$$

$$= \frac{e\hbar}{2mL}\left\{ e^{i(k_m + k_m)x} (k_m - k_m) J_{mm'} + \sum_{m=1}^{N_R} (R'_{mm})^* J_{mm'}\, 2k_m \right\}$$

$$= \frac{e\hbar}{2mL}\left\{ e^{i(k_m + k_m)x} \underbrace{(k_m - k_m)}_{=0} J_{mm'} + 2\sqrt{k_m}\sqrt{k_{m'}} (r'^+ t)_{mm'} \right\}$$

cross-terms $\alpha = L,\ \alpha' = R \qquad x \in R$

$$i^{(x)}_{nLE\,n'RE} = \frac{e\hbar}{2im}\left\{ \int dy \left[\sum_{m=1}^{N_R} J_{mn}\, \psi^+_{mRE} \right]^* \partial_x \left[\psi^-_{n'RE} + \sum_{m'=1}^{N_R} R'_{m'n'}\, \psi^+_{m'RE} \right] \right.$$

$$\left. - \int dy\, \partial_x \left[\sum_{m=1}^{N_R} J_{mn}\, \psi^+_{mRE} \right]^* \left[\psi^-_{n'RE} + \sum_{m'=1}^{N_R} R'_{m'n'}\, \psi^+_{m'RE} \right] \right\}$$

$$= \frac{e\hbar}{2im} \left[e^{-ik_{n'}x} J^*_{n'n}\, e^{-ik_{n'}x} (-ik_{n'}) + \sum_{m=1}^{N_R} J^*_{mn} R'_{mn'}\, e^{-ik_m x} e^{ik_m x} (ik_m) \right.$$

$$\left. - e^{-ik_{n'}x} J^*_{n'n} (-ik_{n'})\, e^{-ik_{n'}x} - \sum_{m=1}^{N_R} J^*_{mn} R'_{mn'}\, e^{-ik_m x} (-ik_m)\, e^{ik_m x} \right]$$

$$= \frac{e\hbar}{m} \sum_{m=1}^{N_R} J^*_{mn} R'_{mn'}\, k_m$$

$$= \frac{e\hbar}{m} (t^+ r')_{nn'} \sqrt{k_n}\sqrt{k_{n'}}$$

cross-terms $\alpha = R$, $\alpha' = L$ $x \in R$

$$i^{(x)}_{mRE\, m'LE} = \frac{e\hbar}{2im} \left\{ \int dy \left[\psi^-_{mRE} + \sum_{m=1}^{N_R} R'_{mm} \psi^+_{mRE} \right]^* \partial_x \sum_{m'=1}^{N_R} J_{m'm'} \psi^+_{m'RE} \right.$$

$$\left. - \int dy\, \partial_x \left[\psi^-_{mRE} + \sum_{m=1}^{N_R} R'_{mm} \psi^+_{mRE} \right]^* \sum_{m'=1}^{N_R} J_{m'm'} \psi^+_{m'RE} \right\}$$

$$= \frac{e\hbar}{2im} \left[e^{ik_m x} J_{mm'} e^{ik_m x} (ik_m) + \sum_{m=1}^{N_R} R'^*_{mm} J_{mm'} e^{-ik_m x} e^{ik_m x} (ik_m) \right.$$

$$\left. - e^{ik_m x} (ik_m) J_{mm'} e^{ik_m x} + \sum_{m=1}^{N_R} R'^*_{mm} J_{mm'} (-ik_m) e^{-ik_m x} e^{ik_m x} \right]$$

$$= \frac{e\hbar}{m} \sum_{m=1}^{N_R} R'^*_{mm} J_{mm'} k_m$$

$$= \frac{e\hbar}{m} \sqrt{k_m} \sqrt{k_{m'}} \, (r'^{+} t)_{mm'}$$