Quantum theory of condensed matter II

Mesoscopic physics (Quantum transport)

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Tue	8:00 - 10:00	9.2.01
Fri	10:00 - 12:00	9.2.01
Fri	12:00 - 14:00	5.0.20

Sheet 2

1. Density of states in reduced dimensions

The density of states (DOS) is a basic quantity useful to characterize a physical system. Fingerprints of the shape of the density of states $D(E)$ are observed both in the equilibrium state and non-equilibrium dynamics. The DOS is defined as:

$$D(E) = \sum_i \delta(E - \epsilon_i), \tag{1}$$

where i denotes the set of quantum numbers characterizing the system. E.g. for free electrons in semiconductors, it is

$$\epsilon_i = \epsilon_{\vec{k}\sigma} = \frac{\hbar^2|\vec{k}|^2}{2m^*}, \tag{2}$$

with m^* the effective mass, $\vec{k}$ and σ the electron wave number and the spin.

1. Using equations (**??**) and (**??**), calculate the DOS for an electron gas in zero (quantum dot), one (quantum wire), two (2DEG), and three (bulk) dimensions. Sketch the function $D(E)$.
2. Do you recognize any relation between dimensionality d, dispersion relation and energy dependence of $D(E)$?
3. What changes in the DOS if the electrons follow a linear dispersion relation instead of the quadratic one given in equation (**??**)?

(6 Points)

2. Two dimensional electron gases in semiconductor heterostructures

The effective conduction band edge of a heterostructure composed of several semiconducting crystals has a complex structure like the one depicted in Fig.**??**. At the interface between the substrate GaAs and the spacer AlGaAs, the conduction band forms an almost perfect triangular quantum well, as sketched in Fig.**??**, which, under specific conditions, hosts a two dimensional electron gas (2DEG).

1. Prove that the conduction band edge forms at the interface between the substrate GaAs and the spacer AlGaAs (i.e. $z \approx 0$) a triangular well given by

$$E_\mathrm{c}(z) = \frac{e^2 N_\mathrm{D} d}{\epsilon^*} z + \Delta E_\mathrm{c} \theta(-z), \tag{3}$$

where N_D is the average density of Si donors in the n-doped AlGaAs layer of thickness d, e is the magnitude of the electronic charge, ϵ^* is the dielectric constant of GaAs. Moreover, ΔE_c is the conduction band discontinuity at the interface of AlGaAs and GaAs and $\theta(z)$ is the Heaviside function. The derivation of Eq. (**??**) rely on the assumption that the only free carriers in the GaAs are all the electrons of the Si donors.
Hint: Take advantage of the global charge neutrality of the system and think in terms of a plane capacitor. Apply Gauß theorem using a cylindrical box with top and bottom lid at the interface and far inside the substrate GaAs and calculate the electric field in the vicinity of $z = 0$.

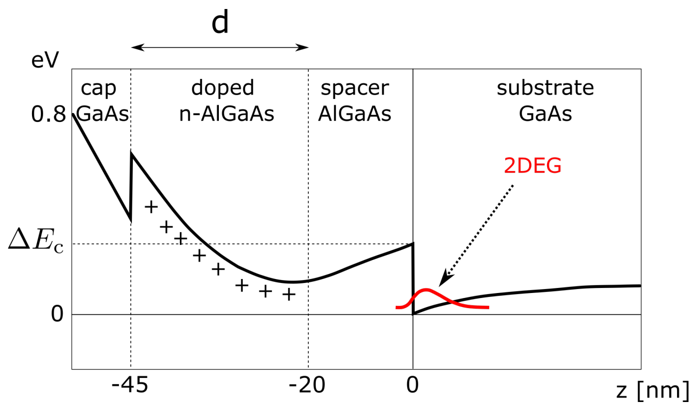

Figure 1: The conduction band edge in a GaAs/AlGaAs heterostructure, with the characteristic triangular well forming at the interface. The donor charges of the n-doped layer are indicated. In red a sketch of the wavefunction for the lowest subband of the 2DEG.

2. The triangular well just calculated in the previous point supports a set of states bound in the z direction with energies ε_n. Assume $\Delta E_c \gg \varepsilon_1$ and solve the Schrdinger equation for the triangular well:

$$\left[-\frac{\hbar^2}{2m^*}\nabla^2 + E_c(z)\right]\psi(x,y,z) = \mathcal{E}\psi(x,y,z) \tag{4}$$

with m^* the effective electronic mass in GaAs. Prove that the energy eigenfunctions read

$$\psi_{n\vec{k}}(x,y,z) = \frac{1}{\mathcal{N}}e^{ik_x x}\, e^{ik_y y}\zeta_n(z) \tag{5}$$

where $\mathcal{N}$ is a normalization factor and $\zeta_n(z)$ can be written in terms of the Airy function:

$$\zeta_n(z) = \mathrm{Ai}\left(\frac{eEz}{\varepsilon_0} + a_n\right) \tag{6}$$

where $E = \frac{eN_{\rm D}d}{\varepsilon^*}$, $\varepsilon_0 = \left[\frac{(\hbar eE)^2}{2m^*}\right]^{\frac{1}{3}}$, and a_n is the $n^{\rm th}$ zero of the Airy function. The first few zeros are:

$$a_1 = -2.338, \qquad a_2 = -4.088, \qquad a_3 = -5.521, \qquad a_4 = -6.787\,.$$

Hint: The Airy function $\mathrm{Ai}(x)$ is one of the two solutions (the integrable one) of the Airy equation

$$\frac{d^2y}{dx^2} = xy$$

studied by the 19th century mathematician W. Airy (1801-1892) to understand the origin of multiple rain-bows. Transform at first Eq. (**??**) into the Airy equation. A variable rescaling $z = \ell\xi$ in Eq. (**??**) is useful to the purpose. The quantization is obtained by imposing $\zeta_n(0) = 0$. Can you justify this point?

3. Prove that the energy eigenvalues associated to $\psi_{n\vec{k}}$ are:

$$\mathcal{E}_{n\vec{k}} = \frac{\hbar^2|\vec{k}|^2}{2m^*} + \varepsilon_n = \frac{\hbar^2|\vec{k}|^2}{2m^*} - \varepsilon_0 a_n$$

thus defining a set of electronic subbands labelled by the quantum number n.

4. Calculate the Fermi energy $\varepsilon_{\rm F}$ corresponding to the electron density of the 2DEG and estimate at which temperature you expect to observe thermal population of the second subband ($n = 2$). Take as parameters $N_{\rm D} = 10^{23}$ m^{-3}, $d = 25$nm, $\epsilon^* = 13\epsilon_0$, $m^* = 0.067m_{\rm e}$ with $m_{\rm e}$ the bare electron mass.

Frohes Schaffen!