Assignments to Condensed Matter Theory I
Sheet 6

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sheet online: http://www-MCG.uni-R.de/teaching/

Problem set: Second quantization II (fermionic gymnastic)

6.1. Fermionic commutation relations

The basis commutation relations for fermion creation and annihilation operators

$$[a, a^\dagger]_+ = 1, \quad [a, a]_+ = 0, \quad a\,|0\rangle = 0$$

where $[A, B]_+ = AB + BA$, $|0\rangle$ the vacuum, and $\dagger$ indicates the Hilbert space adjoint.

(a) From these, determine all normalized eigenstates $|n\rangle$ of $a^\dagger a$, and show that they have the following properties,

$$\begin{aligned} a^\dagger a\,|n\rangle &= n\,|n\rangle\,, \quad n = 0, 1 \\ a\,|1\rangle &= |0\rangle\,, \\ a^\dagger\,|0\rangle &= |1\rangle\,, \\ \langle n, m\rangle &= \delta_{nm} \end{aligned}$$

(b) Compute $F = -k_\mathrm{B}T \ln Z$ with

$$Z = \mathrm{Tr}\left\{\exp\left(-\frac{\hbar\omega}{k_\mathrm{B}T}a^\dagger a\right)\right\} = \sum_n \left\langle n\middle|\exp\left(-\frac{\hbar\omega}{k_\mathrm{B}T}a^\dagger a\right)\middle|n\right\rangle$$

6.2. [Kür] Yukawa-correlated fermions

Consider a system of fermions created by the field $\psi^\dagger(\mathbf{r})$ interacting under the Yukawa potential

$$V_\mathrm{Y}(\mathbf{r}) \equiv V_\mathrm{Y}(r) = \frac{A}{r}\exp(-\alpha r) \qquad (\alpha > 0)$$

(a) Write the Hamiltonian in second quantized form, using the position basis.

(b) Write the Hamiltonian in second quantized notation in the momentum basis, where

$$c^\dagger_\mathbf{k} = \int \mathrm{d}^3r\, \psi^\dagger(\mathbf{r})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

Hint: You will find it helpful to derive the Fourier representation

$$V_{\mathrm{Y}}(\mathbf{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}} \frac{4\pi A}{(q^2+\alpha^2)}.$$

6.3. Calculating with fermion operators

(a) Similarly to exercise 3.3, simplify the following expressions involving the fermionic operators a, and $a^\dagger$

$$g_1(\alpha; a, a^\dagger) = \mathrm{e}^{-\alpha a^\dagger} a \mathrm{e}^{\alpha a^\dagger}, \qquad h_1(\alpha; a, a^\dagger) = \mathrm{e}^{-\alpha a} a^\dagger \mathrm{e}^{\alpha a},$$
$$g_2(\alpha; a, a^\dagger) = \mathrm{e}^{-(\alpha^* a^\dagger - \alpha a)} a \mathrm{e}^{(\alpha^* a^\dagger - \alpha a)}, \quad h_2(\alpha; a, a^\dagger) = \mathrm{e}^{-(\alpha^* a^\dagger - \alpha a)} a^\dagger \mathrm{e}^{(\alpha^* a^\dagger - \alpha a)},$$
$$g_3(\alpha; a, a^\dagger) = \mathrm{e}^{-\alpha a^\dagger a} a \mathrm{e}^{\alpha a^\dagger a}, \qquad h_3(\alpha; a, a^\dagger) = \mathrm{e}^{-\alpha a^\dagger a} a^\dagger \mathrm{e}^{\alpha a^\dagger a}.$$

(b) Show that the operators s^+, s^-, and s^z, defined in terms of the fermionic operators $a^\dagger_{\uparrow,\downarrow}$ and $a_{\uparrow,\downarrow}$ (the indices $\uparrow$, $\downarrow$ characterize the electron possible spin states)

$$\begin{aligned} s^+ &= \hbar a^\dagger_\uparrow a_\downarrow \\ s^- &= \hbar a^\dagger_\downarrow a_\uparrow \\ s^z &= \frac{\hbar}{2}\left(a^\dagger_\uparrow a_\uparrow - a^\dagger_\downarrow a_\downarrow\right) \end{aligned}$$

satisfy the commutation relations for spin components, that is

$$\begin{aligned} \left[s^+, s^-\right] &= 2\hbar s^z \\ \left[s^\pm, s^z\right] &= \mp\hbar s^\pm \end{aligned}$$

Hint: Use the fact that

$$\begin{aligned} [A, BC] &= [A, B]\, C + B\, [A, C] \\ [AB, C] &= A\, [B, C]_+ - [A, C]_+\, B \end{aligned}$$