University of Regensburg Summer Semester 2007

# Quantum Theory of Condensed Matter

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## Sheet 2

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## 1. Cubic corrections to the q-harmonic oscillator

Calculate the correction to the frequency of an oscillator in its ground state due to a cubic anharmonicity. Cubic anharmonicities arise *e.g.* by expanding a lattice potential $V(\hat{x})$ beyond the harmonic terms.

1. Starting from a Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ in terms of position and momentum operators, prove that its second quantization form reads:

$$\hat{H} = \hbar\omega_0 \left( a^\dagger a + \frac{1}{2} \right) + \Delta(a^\dagger + a)^3 \; .$$

2. Treat the anharmonicity by expanding the third order term and bring it into normal order (*i.e.* Bring all the "dagger to the left" making use of the bosonic commutator relations). Check the result:

$$(a^\dagger + a)^3 = a^{\dagger 3} + 3a^{\dagger 2}a + 3(a^\dagger + a) + 3a^\dagger a^2 + a^3.$$

3. Sometimes quantum mechanical operators can be replaced by their mean value without loosing too much of the underlying physics (mean field approximation). Replace pairs of operators by their thermal expectation value and neglect mean values of operators that do not conserve number of particles. You are then left only with the thermal average of the number operator that is called the Bose function and reads:

$$\langle a^\dagger a \rangle = n_B(T) \equiv \frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1},$$

   and $\beta = 1/k_B T$.

4. After the point 1. and 2. you obtain a quadratic Hamiltonian that we solve by means of a symmetry operation. Remember that the spectrum of the Hamiltonian is invariant under spatial translation and that a translation of a quantum mechanical operator $\hat{O}$ by the length $\alpha$ is represented by:

$$\mathcal{T}_\alpha(\hat{O}) = \exp\left(\mathrm{i}\frac{\alpha\hat{p}}{\hbar}\right) \hat{O} \exp\left(-\mathrm{i}\frac{\alpha\hat{p}}{\hbar}\right).$$

   For example, $\mathcal{T}_\alpha(\hat{x}) = \hat{x} + \alpha$. Calculate how the translation operation acts on the bosonic creator operator by proving that:

$$\exp\left(\mathrm{i}\frac{\alpha\hat{p}}{\hbar}\right) a^\dagger \exp\left(-\mathrm{i}\frac{\alpha\hat{p}}{\hbar}\right) = a^\dagger + \alpha\sqrt{\frac{m\omega}{2\hbar}}.$$

5. How does it go with the annihilator?

6. Apply a translation operation to the linearized Hamiltonian and choose the constant $\alpha$ so to cancel the linear term. Which is the spectrum of the resulting Hamiltonian?

## 2. Calculating with fermion operators

The basis commutation relations for fermion creation and annihilation operators are

$$[a, a^\dagger]_+ = 1, \quad [a, a]_+ = 0, \quad a|0\rangle = 0,$$

where $[A, B]_+ = AB + BA$, $|0\rangle$ the vacuum, and $\dagger$ indicates the Hilbert space adjoint. (Turn the page...)

1. Similarly to exercise 2 of the Sheet 1, simplify the following expressions involving the fermionic operators $a$, and $a^\dagger$

$$g(\alpha; a, a^\dagger) = \mathrm{e}^{-(\alpha^* a^\dagger - \alpha a)} a \mathrm{e}^{(\alpha^* a^\dagger - \alpha a)}, \quad h(\alpha; a, a^\dagger) = \mathrm{e}^{-\alpha a^\dagger a} a \mathrm{e}^{\alpha a^\dagger a}.$$

## Frohes Schaffen!