University of Regensburg | Summer Term 2013

Applications of Group Theory

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Lectures 9.2.01, Mondays, 14:15
Exercises 7.1.21, Fridays, 10:15

Sheet 1

1. Finite vs. infinite groups

A group is generally defined as a set of elements $\mathcal{G}$ and a binary composition $(\cdot)$ satisfying the four following properties:

i) Closure of $\mathcal{G}$ with respect to the binary composition;

ii) Existence of the identity element;

iii) Validity of the associative law;

iv) Existence of the inverse of each element of $\mathcal{G}$ inside the set $\mathcal{G}$ itself.

In reality some care should be taken with the order of the group.

1. Prove that, for a group of finite order, the property iv) can be derived from the other three. Hint: Start by proving that $\forall g \in \mathcal{G}, \exists n \in \mathbb{N}^+ : g^n = E$, where E is the identity element.
2. Consider the set of positive integers $\{p\}$ with the addition operation. Is it a group? If you add 0 ? What about the set of all (positive and negative) integer numbers with the same operation?

2. Cyclic groups

Consider the sequence g, g^2, ... and assume that $g^{n+1} = g$ with $n \in \mathbb{N}$.

1. Verify that $\{g, g^2, \ldots, g^n\}$ is a group: it is a cyclic group.
2. Show that cyclic groups are abelian.
3. Show that for a finite cyclic group the existence of the inverse of each element is guaranteed.
4. Show that $\omega = \exp(-2\pi \mathrm{i}/n)$ generates a cyclic group of order n, when binary composition is defined to be the multiplication of complex numbers.

3. Groups of order 4

There are only two groups of order 4 that are not isomorphous and so have different multiplication tables. Derive the multiplication tables of these two groups, $\mathcal{G}_4^1$ and $\mathcal{G}_4^2$. Hints: First derive the multiplication table of the cyclic group of order 4. Call this group $\mathcal{G}_4^1$. How many elements of $\mathcal{G}_4^1$ are equal to their inverse? Now try to construct further groups in which a different number of elements are equal to their own inverse. Observe the rearrangement theorem. It is a bit like sudoku!

Frohes Schaffen!