Quantum Theory of Condensed Matter I

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5.1.01 Mondays 10:15
9.2.01 Tuesdays 12:15

Sheet 11

1. Quasi particles density of states

Consider the BCS Hamiltonian in the mean field approximation:

$$H_{BCS}^{MF} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta\, c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \sum_{\mathbf{k}} \Delta^*\, c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \tag{1}$$

where the single particle energy is measured starting from the chemical potential: $\xi_{\mathbf{k}} \equiv \frac{\hbar^2 k^2}{2m} - \mu$. As seen in the lecture the Hamiltonian (1) is diagonalized by the Bogoliubov transformation:

$$\begin{pmatrix} \alpha_{\mathbf{k}\uparrow} \\ \alpha^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u^*_{\mathbf{k}} & v_{\mathbf{k}} \\ -v^*_{\mathbf{k}} & u_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{2}$$

with the values

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \qquad |v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \tag{3}$$

where

$$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + \Delta^2} \tag{4}$$

1. Consider the Bogoliubov transformation applied to a normal system ($\Delta = 0$). How does the Hamiltonian look like in terms of the operators $\alpha_{\mathbf{k}\sigma}$ and $\alpha^\dagger_{\mathbf{k}\sigma}$? In which sense the operator $\alpha^\dagger_{\mathbf{k}\sigma}$ creates a quasi particle excitation? Which is the spectrum of these excitations as a function of $\xi_{\mathbf{k}}$

   (3 Points)

2. The Bogoliubov transformation brings the Hamiltonian (1) into the form:

$$\sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}\sigma} \alpha_{\mathbf{k}\sigma} + \text{constant}, \tag{5}$$

   where $E_{\mathbf{k}}$ is given above. Calculate the density of states $d_s(\epsilon)$ for the quasi-particle excitations of a superconductor described by the BSC Hamiltonian (1). Prove that, in the limit $\epsilon, \Delta \ll \mu$ the following relation holds:

$$\frac{d_s(\epsilon)}{d_n(0)} = \theta(\epsilon - |\Delta|) \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} \tag{6}$$

   where $d_n(0)$ is the density of states of the zero energy quasi-particle excitations of a normal metal ($\Delta = 0$).
   Hint: Remember the definition of density of states: $d(\epsilon) = \frac{1}{V}\sum_\alpha \delta(\epsilon - \epsilon_\alpha)$.

   (3 Points)

2. Average particle number of a superconductor

Consider the celebrated variational BCS ground state:

$$|\psi_{BCS}\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow})|0\rangle \tag{7}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are implicitly given in the previous exercise if we assume them to be real, $|0\rangle$ is the vacuum state. Show that the average electron number associated to the BCS ground state reads:

$$\langle N \rangle = \langle \psi_{BCS} | \hat{N} | \psi_{BCS} \rangle = 2 \sum_{\mathbf{q}} |v_{\mathbf{q}}|^2 \tag{8}$$

(3 Points)

Frohes Schaffen!