University of Regensburg | Summer Term 2016

Applications of Group Theory

PD Dr. Andrea Donarini

Lectures: H33, Mondays, 14:15; H34, Thursdays, 14:15

Exercises: 5.0.21, Wednesdays, 13:15

Sheet 1

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1. Finite vs. infinite groups

A group is generally defined as a set of elements $\mathcal{G}$ and a binary composition $(\cdot)$ satisfying the four following properties: i) Closure of $\mathcal{G}$ with respect to the binary composition; ii) Existence of the identity element; iii) Validity of the associative law; iv) Existence of the inverse of each element of $\mathcal{G}$ inside the set $\mathcal{G}$ itself. Some care, though, should be taken with the order of the group.

1. Consider the set of positive integers $\{p\}$ with the addition operation. Is it a group? If you add 0 ? What about the set of all (positive and negative) integer numbers with the same operation?
2. Prove that, for a group of finite order, the property iv) can be derived from the other three. Hint: Start by proving that $\forall g \in \mathcal{G}, \exists n \in \mathbb{N}^+ : g^n = E$, where E is the identity element.

2. Cyclic groups

Consider an element (e.g. a symmetry operation) g and a binary composition $\cdot$ that allows you to construct the sequence $g_1 \equiv g$, $g_2 \equiv g \cdot g$, $g_3 \equiv g \cdot g \cdot g$, and so on. Further assume that $g_{n+1} = g$ with $n \in \mathbb{N}$.

1. Verify that the set $\{g_1, g_2, \ldots, g_n\}$ with the operation $\cdot$ is a group by explicitly verifying the closure, the existence of the identity, of the inverse and the associative law. These type of groups, generated by repeated multiplication of the same generator element are called cyclic groups.
2. Show that cyclic groups are abelian.
3. The set of all complex numbers with the usual complex number multiplication is a group. Is it containing a cyclic subgroup of order 2? And one of order $n \in \mathbb{N}$?

3. Groups of order 4

There are only two groups of order 4 that are not isomorphous and so have different multiplication tables. Derive the multiplication tables of these two groups, $\mathcal{G}_4^1$ and $\mathcal{G}_4^2$. Hints: First derive the multiplication table of the cyclic group of order 4. Call this group $\mathcal{G}_4^1$. How many elements of $\mathcal{G}_4^1$ are equal to their inverse? Now try to construct further groups in which a different number of elements are equal to their own inverse. Remember to fulfil the rearrangement theorem. It is a bit like sudoku!

Frohes Schaffen!