4. Basics of semiclassical transport: Boltzmann equation

In Boltzmann formalism the electron kinetics is described in terms of the distribution function in the phase space: $f(\bar{r}, \bar{k}, t)$

This is essentially a classical description; in quantum mechanics $\Delta x \cdot \Delta k_x \geq 1/2 \Rightarrow \bar{r}$ and $\bar{k}$ cannot be fixed simultaneously. Such a classical description is applicable (with electrons represented by wave packets) as long as external fields are smooth on the scale $\lambda_F \equiv 2\pi / k_F$.

Liouville's theorem (follows from Hamilton's equations)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\bar{r}} \frac{\partial f}{\partial \bar{r}} + \dot{\bar{k}} \frac{\partial f}{\partial \bar{k}} = 0$$

$$\dot{\bar{r}} = \bar{v} = \frac{1}{\hbar} \frac{\partial \epsilon(k)}{\partial \bar{k}} \stackrel{*}{=} \frac{\hbar \bar{k}}{m} \quad \left(* \text{ for } \epsilon = \frac{\hbar^2 k^2}{2m}\right)$$

$$\hbar \dot{\bar{k}} = \bar{F} = (-e)(\bar{E} + \bar{v} \times \bar{B})$$

In addition, there are scattering processes which can not be described classically, including electron–electron and electron–phonon processes and elastic scattering off impurities.

Collision (Stoß)term: $\frac{df}{dt} = St\{f\} \equiv \left(\frac{df}{dt}\right)_{coll}$

$$\boxed{\frac{\partial f}{\partial t} + \bar{v} \frac{\partial f}{\partial \bar{r}} + \frac{\bar{F}}{\hbar} \frac{\partial f}{\partial \bar{k}} = \left(\frac{df}{dt}\right)_{coll}} \qquad (4.1)$$

Boltzmann equation

At low temperatures elastic scattering dominates

$W_{kk'}$ – probability of elastic scattering from state k to k'

Weak impurities $\Longrightarrow$ time-dependent perturbation theory yields

$$W_{\bar{k}\bar{k}'} = \frac{2\pi}{\hbar} \underbrace{n_i}_{\text{density of impurities}} \underbrace{\delta(\varepsilon_{\bar{k}} - \varepsilon_{\bar{k}'})}_{\text{conservation of energy}} |\langle \bar{k} | \underbrace{U}_{\text{potential of single impurity}} | k' \rangle|^2 \qquad (4.2)$$

In what follows, we also use $W_{\bar{k}\bar{k}'} \equiv \delta(\varepsilon_{\bar{k}} - \varepsilon_{\bar{k}'}) \underbrace{\gamma_{\bar{k}\bar{k}'}}_{|k| = |k'|}$

Using $W_{kk'}$, we can write

$$\left(\frac{df(k)}{dt}\right)_{coll} = -\int \frac{d\bar{k}'}{(2\pi)^d} \left\{ W_{\bar{k}\bar{k}'} f(k)[1 - f(k')] - W_{k'k} f(k')[1 - f(k)] \right\}$$

Pauli principle

$$W_{kk'} \overset{(4.2)}{=} W_{k'k}$$

$$\Longrightarrow \left(\frac{df(k)}{dt}\right)_{coll} = -\int \frac{d\bar{k}'}{(2\pi)^d} W_{\bar{k}\bar{k}'} [f(\bar{k}) - f(\bar{k}')] \qquad (4.3)$$

Linear-response dc conductivity

Consider electron gas in the presence of homogeneous, static, and infinitesimaly weak electric field $\vec{E} \parallel \hat{x}$, $|E| \to 0$

$$\cancel{\frac{\partial f}{\partial t}} + \bar{v} \cancel{\frac{\partial f}{\partial \bar{r}}} - eE \frac{\partial f}{\partial \hbar k} = -\int \frac{d\bar{k}'}{(2\pi)^d} W_{\bar{k}\bar{k}'} [f(\bar{k}') - f(\bar{k})] \qquad (4.4)$$

E does not depend on $\bar{r}$ and t $\Longrightarrow$ $f(\bar{r}, \bar{k}, t) = \text{const}(\bar{r}, t)$

$$\vec{E}=0 \Rightarrow f = f_0(\epsilon_k) = \frac{1}{e^{\beta(\epsilon_k-\mu)}+1}$$ equilibrium Fermi-Dirac distribution

$$E \neq 0 \Rightarrow f = f_0 + \delta f, \quad \delta f = O(E)$$ - linear-in-$\vec{E}$ correction (linear response)

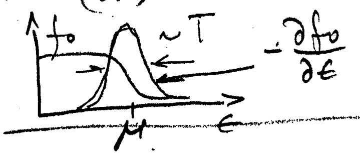
To linear order, $\frac{1}{\hbar}\vec{E}\frac{\partial f}{\partial \vec{k}} \simeq \frac{1}{\hbar}\vec{E}\frac{\partial f_0}{\partial \vec{k}} = \frac{1}{\hbar}\vec{E}\,\frac{\partial \epsilon}{\partial \vec{k}}\cdot\frac{\partial f_0}{\partial \epsilon_k} = \vec{E}\cdot\vec{v}\,\frac{\partial f_0}{\partial \epsilon_k}$ (4.5)

$\Rightarrow$ Look for solution of (4.4) in the form

$$\delta f(\vec{k}) = \left(-\frac{\partial f_0}{\partial \epsilon_k}\right) F(n_{\vec{k}}) \qquad (4.6)$$

$n_{\vec{k}}$ parametrizes the surface $\epsilon_k \propto k^2 = \text{const}$ 3D: $n_{\vec{k}} \equiv \{\theta, \varphi\}$; 2D: $n_{\vec{k}} \equiv \varphi$

$$\int \frac{d\vec{k}'}{(2\pi)^d} = \int d\epsilon_{k'}\,\nu(\epsilon_{k'}) \int d\Omega_{k'}; \quad d\Omega_k = \frac{d\varphi}{2\pi}\ (2D), \quad d\Omega_k = \frac{\sin\theta\, d\theta\, d\varphi}{4\pi}\ (3D) \qquad (4.7)$$



(B4): $$-e\vec{E}\cdot\vec{v} = \nu_k \int d\Omega_{k'}\,[F(n_{\vec{k}}) - F(n_{\vec{k}'})]\,\gamma_{\vec{k}\vec{k}'} \qquad (4.8)$$

Solve for 2D:

$$-eEv\cos\varphi = \nu \int \frac{d\varphi'}{2\pi}\,[F(\varphi) - F(\varphi')]\,\gamma_k(\varphi-\varphi') \qquad (4.9)$$

where $v = \hbar|k|/m$.

(!) $\gamma_{\vec{k}\vec{k}'} = \gamma_{|k|}(\varphi-\varphi')$ - scattering rate depends only on $|k|=|k'|$ and on $\varphi-\varphi'$ — no preferred direction in disorder-averaged system

Solution has the form $F(\varphi) = -eEv\cos\varphi \cdot \tau$ (4.10)

τ - transport scattering time to be found below

$$\varphi'' = \varphi' - \varphi$$

$$\oint d\varphi' (\cos\varphi - \cos\varphi')\,\gamma(\varphi - \varphi') = \cos\varphi \oint d\varphi''\,\gamma(\varphi'') \qquad (4.11)$$

$$- \oint d\varphi'' \cos(\varphi'' + \varphi)\,\gamma(\varphi'') = \cos\varphi \int d\varphi''\,\gamma(\varphi'')(1 - \cos\varphi'')$$

$\Rightarrow$ (4.9) with $F(\varphi)$ given by (4.10) reads

$$-eEv\cos\varphi = \nu \int \frac{d\varphi''}{2\pi} \left(-eEv\tau\cos\varphi\right) \gamma(\varphi'')(1 - \cos\varphi'')$$

$$\Rightarrow \quad \tau_k^{-1} = \nu \int \frac{d\varphi}{2\pi}\,\gamma(\varphi)(1 - \cos\varphi)$$ – transport relaxation time (4.12)

Current density $\quad \vec{j} = \int \frac{d\bar{k}}{(2\pi)^d} (-e)\,\bar{v}_{\bar{k}}\, f(\bar{k}) \qquad (4.13)$

- Since $\int d\Omega_{\bar{k}}\, \bar{v}_{\bar{k}} = 0$, f_0 yields $\vec{j} = 0$ (no dc current in equilibrium)

- 2D, $E \parallel x$, $T \to 0$

$$j_x = \int \frac{d^2k}{(2\pi)^2} (-e) v_k \cos\varphi\, \delta f(\bar{k}) \longleftarrow (4.6),(4.10)$$

$$= \int d\epsilon_k\, \nu(\epsilon_k) \int \frac{d\varphi}{2\pi} (-e) v_k \cos\varphi \left(-\frac{\partial f_0}{\partial \epsilon_k}\right) eE v_k \cos\varphi\, \tau_k$$

$$= -e^2 \int d\epsilon_k \frac{\partial f_0}{\partial \epsilon_k}\, \nu(\epsilon_k) \frac{v_k^2}{2} \tau_k \overset{T \ll \mu}{\simeq} e^2 \nu(\mu) \frac{v_F^2}{2} \tau_{k_F} E = \sigma E$$

$\boxed{\sigma = e^2 \nu \frac{v_F^2}{2} \tau}$ – Drude formula for dc conductivity in 2D (4.14)

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d dimensions $\quad F(n_{\bar{k}}) = -e\,\bar{E} \cdot \bar{v}\,\tau$ – solution to B. equation

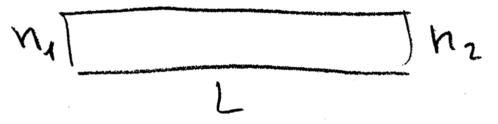
$$\tau_k^{-1} = \nu \int d\Omega'\, \gamma_{\bar{k}\bar{k}'} \left(1 - \frac{\bar{k} \cdot \bar{k}'}{k^2}\right)$$

$$\boxed{\sigma = -e^2 \int d\epsilon_k \frac{\partial f_0}{\partial \epsilon_k}\, \nu(\epsilon_k) \frac{v_k^2}{d} \tau_k \overset{T \ll \mu}{\simeq} e^2 \nu(\mu) \frac{v_F^2}{d} \tau_{k_F}} \qquad (4.15)$$

Drude formula

Diffusion, Einstein relation (2D, $T \to 0$)

Consider situation when no electric field is applied, but the is a small gradient of concentration



$$\frac{n_1 - n_2}{L} = \nabla n \parallel \hat{x} \qquad |\nabla n| = \text{const} \to 0$$

BE (B1): $\quad \bar{v}\frac{\partial f}{\partial r} = \left(\frac{df}{dt}\right)_{coll} \qquad (4.16)$

$$f = f_0\left[\epsilon_k - \mu(r)\right] + \delta f, \qquad \mu = \mu_0 + |\nabla\mu|\, x \overset{T=0}{=} \mu_0 + \frac{x}{\nu}|\nabla n|$$

$$\bar{v}\frac{\partial f}{\partial r} \simeq \bar{v}\frac{\partial}{\partial \bar{r}} f_0(\epsilon_k - \mu(r)) = -\bar{v}\frac{\partial f_0}{\partial \epsilon_k}\nabla\mu(r) \simeq -v\cos\varphi\frac{\partial f_0}{\partial \epsilon_k}|\nabla\mu|$$

$$\boxed{-\frac{v_F}{\nu}|\nabla n|\cos\varphi = \nu\int\frac{d\varphi'}{2\pi}\left[F(\varphi) - F(\varphi')\right]\gamma(\varphi - \varphi') \quad (4.17)}$$

Same equation as before up to $-e\bar{E} \to -\frac{1}{\nu}\nabla n$

$$\Rightarrow F(\varphi) = -\frac{1}{\nu}\nabla n\, v_F\cos\varphi \cdot \tau$$

$$j_x = e\nu\frac{v_F^2}{2}\tau\frac{1}{\nu}\nabla n = e D\nabla n, \quad D = \frac{v_F^2\tau}{2}$$

d-dimensions: $\quad \bar{j} = e D\bar{\nabla} n, \quad D = \frac{v_F^2\tau}{d} \qquad (4.18)$

$\bar{j}$: diffusion current; D: diffusion coefficient

$$\boxed{\sigma = e^2 \nu D} \quad \text{– Einstein relation}$$

General version: $\quad \sigma = e^2\left(\frac{\partial n}{\partial \mu}\right) D \qquad (4.19)$

Compressibility $\chi = \frac{\partial n_e}{\partial \mu} = \begin{cases} \nu, & T \ll \mu \quad \text{(degenerate Fermi gas, quantum)} \\ \frac{n_e}{kT}, & T \ll |\mu|,\ \mu < 0 \quad \text{(non-degenerate Boltzmann gas, classical)} \end{cases}$

($\to$ exercise this week)

AC conductivity

$$\frac{\partial f}{\partial t} - e\vec{E}(t)\frac{\partial f}{\partial \hbar \vec{k}} = -\int \frac{dk'}{(2\pi)^d} W_{\vec{k}\vec{k}'}\left[f(\vec{k}) - f(\vec{k}')\right]$$

$$E(t) = E e^{-i\omega t} \xrightarrow{\text{linear}} \text{correction } \delta f e^{-i\omega t} \quad \left[\frac{\partial}{\partial t} \to -i\omega\right]$$

2D: $\delta f(k) = \left(-\frac{\partial f_0}{\partial \epsilon_k}\right) F(\varphi)$

$$\nu \int \frac{d\varphi'}{2\pi}\left[F(\varphi) - F(\varphi')\right]\gamma(\varphi-\varphi') - i\omega F(\varphi) = -eE v_F \cos\varphi \quad (4.20)$$

$$\Rightarrow F(\varphi) = -eE v_F \cos\varphi\, \tilde{\tau}, \quad \tilde{\tau} = \frac{\tau}{1-i\omega\tau}$$

$$\boxed{\sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau}, \quad \sigma_0 = e^2\nu\frac{v_F^2}{2}\tau} \quad (4.21)$$ Drude-Lorentz formula

Time representation

$$(4.20) \to \frac{\partial F(\varphi,t)}{\partial t} + \nu\int\frac{d\varphi'}{2\pi}\left[F(\varphi,t) - F(\varphi',t)\right]\gamma(\varphi-\varphi') = -eE(t)v_F\cos\varphi \quad (4.22)$$

$$F(\varphi,t) = -e v_F \cos\varphi\, \tilde{F}(t) \to \frac{\partial \tilde{F}}{\partial t} + \frac{1}{\tau}\tilde{F} = E(t) \quad (4.23)$$

$$\Rightarrow \tilde{F}(t) = \int_{-\infty}^{t} dt'\, e^{-\frac{t-t'}{\tau}} E(t')$$

$$j(t) = -e\nu v_F\int \cos\varphi\, F(\varphi,t)\,d\varphi = \frac{e^2\nu v_F^2}{2}\int_{-\infty}^{t} dt' \underbrace{e^{-\frac{t-t'}{\tau}}}_{\text{Memory function}} E(t')$$

$$\equiv \int_{-\infty}^{t} \sigma(\underbrace{t-t'}_{>0})E(t')dt' \text{ — retarded response}$$

$$\sigma(t) = \frac{e^2\nu v_F^2}{2} e^{-t/\tau}\underbrace{\theta(t)}_{t>0} \leftrightarrow \sigma(\omega) = \int_{\underbrace{-\infty}_{\text{can be replaced by } 0}}^{\infty} dt\, e^{i\omega t}\sigma(t) \quad (4.24)$$

Classical analogue of the Kubo formula

$$\sigma(t) = e^2\nu D(t), \quad D(t) = \langle v_x(0)v_x(t)\rangle \text{ — average over ensemble} \quad (4.25)$$

To show this, we take a test particle at Fermi surface with $\vec{V}(t=0) = \vec{V}_0 \leftrightarrow \{V_F, \varphi_0\} \Rightarrow F(\varphi, t=0) = 2\pi\delta(\varphi-\varphi_0)$ (4.26)

and calculate $\langle V_x(t)\rangle = V_F\int F(\varphi,t)\cos\varphi \frac{d\varphi}{2\pi}$ (4.27)
(exercise)

Part II

Many-body aspects

CHAPTER 5 SECOND QUANTIZATION

Aim of this chapter is to introduce the method of second quantization i.e. a formulation based on the algebra of the ladder operators $\hat{a}, \hat{a}^+$ which is convenient to treat problems where several quantum particles are involved (many-body problems).

Motivation for the II quantization formalism:

- Quantum mechanical <u>indistinguishability</u> of identical particles requires symmetrization of the many-body wave function. In I quantization this leads to formal complications, especially in presence of interactions

- I quantization is tailored for problems with fixed particle number N.

$\boxed{5.1}$ Indistinguishability of particles

Classically indistinguishability is taken into account by adding a prefactor $\frac{1}{N!}$ in the counting of microstates. However, indistinguishability of particles has more profound consequences on the form of the many-body wave functions/states.
Specifically, let us consider the N-particles Hamiltonian $\hat{H}_N$, with the associated eigenstate $|\psi\rangle$:

$$\hat{H}_N |\psi\rangle = E_N |\psi\rangle \tag{5.1}$$

and a complete, orthonormal single particle basis $\{|\nu\rangle\}$.

In general, any N-particle state can be written as a linear combination of products of N single-particle states:

$$|\Psi\rangle = \sum_{\nu_1 \dots \nu_N} A_{\nu_1 \dots \nu_N} |\nu_1\rangle \otimes |\nu_2\rangle \otimes \dots \otimes |\nu_N\rangle \qquad (5.2)$$

where $|\nu_j\rangle$ $j = 1 \dots N$ is the quantum state of the j^{th} particle.

For example: $N = 2$

$$|\Psi\rangle = \sum_{\nu_1 \nu_2} A_{\nu_1 \nu_2} |\nu_1\rangle \otimes |\nu_2\rangle$$

Let's introduce the permutation operator $\hat{P}_{12}$ which exchange particle 1 with particle 2.

$$\hat{P}_{12} |\Psi\rangle = \sum_{\nu_1 \nu_2} A_{\nu_1 \nu_2} \hat{P}_{12} |\nu_1\rangle \otimes |\nu_2\rangle = \sum_{\nu_1 \nu_2} A_{\nu_1 \nu_2} |\nu_2\rangle \otimes |\nu_1\rangle \qquad (5.3)$$

$$= \sum_{\nu_1 \nu_2} A_{\nu_2 \nu_1} |\nu_1\rangle \otimes |\nu_2\rangle ,$$

where in the last equality we have simply relabelled the indices. Because of indistinguishability $|\Psi\rangle$ and $\hat{P}_{12}|\Psi\rangle$ must represent the same quantum state. $\Rightarrow$

$$\hat{P}_{12} |\Psi\rangle = \Theta |\Psi\rangle$$

Moreover $\hat{P}_{12}^2 = \mathbb{1} \Rightarrow \Theta^2 = 1 \Rightarrow \Theta = \pm 1$. Given the orthonormality of $\{|\nu\rangle\}$ this implies:

$$A_{\nu_1 \nu_2} = \pm A_{\nu_2 \nu_1} \qquad (5.4)$$

Let us restrict for simplicity the single particle basis to $\{|\nu\rangle\} = \{|a\rangle, |b\rangle\}$

- $\boxed{\Theta = 1}$ $\quad |\Psi\rangle = A_{aa} |a\rangle \otimes |a\rangle + A_{bb} |b\rangle \otimes |b\rangle + A_{ab} \left(|a\rangle \otimes |b\rangle + |b\rangle \otimes |a\rangle \right) \quad (5.5)$

i.e. the state is symmetric upon exchange of the role of particle 1 and particle 2.

• $\boxed{\theta = -1}$ (5.4) $\Rightarrow$ $A_{\nu_j \nu_j} = 0$

hence
$$|\psi\rangle = A_{ab}\left(|a\rangle \otimes |b\rangle - |b\rangle \otimes |a\rangle\right) \qquad (5.6)$$
i.e the state is antisymmetric upon exchange of the role of particle 1 and particle 2.

The many-body wave function can only be

$$\begin{cases} \text{symmetric} & \Leftrightarrow \theta = +1 \\ \text{antisymmetric} & \Leftrightarrow \theta = -1 \end{cases} \quad \text{under } \underline{\text{PERMUTATIONS}} \qquad (5.7)$$

upon exchange of particles (well, there are anyons in 2D associated to braiding groups, instead of permutations).

The property (5.7) is closely related to the spin of the considered particles (or quasiparticles).

$$\begin{cases} \text{integer spin} & \leftrightarrow \text{"bosons"} \\ \text{half-integer spin} & \leftrightarrow \text{"fermions"} \end{cases}$$

<u>Notes</u>:
- This property is independent of whether the particles are independent or interacting with each other
- To construct a properly symmetrized/antisymmetrized wave function is very cumbersome in Hilbert spaces with many particles in 1 quant.
- Fermions: electrons, neutrons, protons, positrons, neutrinos, bound states of an odd number of fermions (e.g. ^3He)

  Bosons: photons, phonons, mesons, Higgs-bosons, gluons or bound states of even number of fermions (e.g. ^4He).

Pauli's principle

From (5.4) it follows that two fermions cannot occupy the same single particle state. In contrast, no such restriction exists for bosons.

- The Pauli exclusion principle is fundamental for understanding the electronic structure of atoms ($\Rightarrow$ the periodic table) and consequently for atomic physics, chemistry, solid state physics. Moreover it is very important also in astrophysics and in the scattering processes in high energy physics.

- The fact that bosons can occupy the same quantum state is at the heart of Bose-Einstein condensation, superconductivity, superfluidity, laser light. The bosonic nature of light is responsible for Planck's radiation law for the electromagnetic field.

Symmetrized/Antisymmetrized basis states

A physically meaningful many-body basis is made of symmetrized/antisymmetrized products of single-particle states. This is accomplished by the symmetrization/antisymmetrization operators $\hat{S}_\pm$.

$$|\psi\rangle_+ = \hat{S}_+ \bigotimes_{j=1}^{N} |\nu_j\rangle = \mathcal{N} \sum_{p \in S_N} \bigotimes_{j=1}^{N} |\nu_{p(j)}\rangle \qquad (5.8)$$

$$|\psi\rangle_- = \hat{S}_- \bigotimes_{j=1}^{N} |\nu_j\rangle = \mathcal{N} \sum_{p \in S_N} \bigotimes_{j=1}^{N} |\nu_{p(j)}\rangle \, \mathrm{sign}(p)$$

where S_N is the group of the $N!$ permutations of N elements and $\mathrm{sign}(p)$ is the sign of the permutation. $\mathrm{sign}(p) = (-1)^{N_i(p)}$ where $N_i(p)$ is the number of inversions in p i.e the number of pairs of elements $\{j_1, j_2\}$ such that

$j_1 < j_2$ but $p(j_1) > p(j_2)$. $\mathcal{N}$ is a normalization factor ensuring $\langle \psi_\pm | \psi_\pm \rangle = 1$

The position and spin representation of the states given in (5.8)

$$\langle \vec{r}_1 \sigma_1, \ldots, \vec{r}_N \sigma_N | \psi_\pm \rangle = \mathcal{N} \begin{vmatrix} \psi_{\nu_1}(\vec{r}_1\sigma_1) & \psi_{\nu_1}(\vec{r}_2\sigma_2) & \ldots & \psi_{\nu_1}(\vec{r}_N\sigma_N) \\ \psi_{\nu_2}(\vec{r}_1\sigma_1) & \psi_{\nu_2}(\vec{r}_2\sigma_2) & \ldots & \psi_{\nu_2}(\vec{r}_N\sigma_N) \\ \vdots & \vdots & & \vdots \\ \psi_{\nu_N}(\vec{r}_1\sigma_1) & \psi_{\nu_N}(\vec{r}_2\sigma_2) & \ldots & \psi_{\nu_N}(\vec{r}_N\sigma_N) \end{vmatrix}_\pm \qquad (5.9)$$

where $| \;|_+ =$ permanent (no sign change) and $| \;|_- =$ determinant

The fermionic wave funct. is commonly referred to as a <u>Slater determinant</u>.

[5.2] <u>Occupation number representation</u>

The symmetrized/antisymmetrized states are fully characterized by the occupation numbers $\{n_\lambda\}$ where λ runs over ALL possible single particle states and n_ν corresponds to the number of times a particular λ appears in the sequence $\nu_1 \ldots \nu_N$ characterising the symmetric/antisymmetric state. To such extend one orders the single-particle states in a sequence $\lambda_1, \lambda_2, \ldots$ where (typically) energetically lower states come first.

One introduces the notation

$$| n_{\lambda_1}, n_{\lambda_2}, \ldots \rangle = | \{ n_{\lambda_i} \} \rangle \qquad (5.10)$$

where, for N-particle states $\sum_i n_{\lambda_i} = N$. Due to the Pauli exclusion principle the number of states available for an N particle system is different for bosons and fermions.

The normalization factor $\mathcal{N}$ introduced in (5.9) is easily written in terms of occupation number representation:

$$\mathcal{N} = \frac{1}{\sqrt{N!}} \frac{1}{\sqrt{\prod_i n_{\lambda_i}!}} \qquad (5.11)$$

As one obtaines by calculation of the scalar product between the permutation of the product of single particle states, keeping in mind that the last are orthonormalized.

5.3 Creation and annihilation operators

An efficient way to construct the totally symmetric/antisymmetric states (5.8) directly connected to their occupation number representation is by means of creation and annihilation operators.

One begins axiomatically with abstract definitions

i) Introduce a normalized reference state $|0\rangle$ called vacuum

ii) Introduce a set of operators $\{\hat{a}_{\lambda_i}\}$ together with their adjoints $\{\hat{a}^+_{\lambda_i}\}$ such that

$$\begin{cases} \hat{a}_{\lambda_i}|0\rangle = 0 \qquad \forall\ \lambda_i \\ \hat{a}^+_{\lambda_i}|0\rangle = |\lambda_i\rangle \\ \text{and} \quad [\hat{a}_{\lambda_i}, \hat{a}^+_{\lambda_j}]_\mp = 1 \qquad [\hat{a}_{\lambda_i}, \hat{a}_{\lambda_j}]_\mp = 0 \end{cases} \tag{5.12}$$

where $[A,B]_\mp = AB \mp BA$

As a consequence of i) and ii) the symmetrized/antisymmetrized states (5.8) can be written as:

$$|\psi\rangle_\pm = \frac{1}{\sqrt{N!\prod_i n_{\lambda_i}!}} \prod_i \left(\hat{a}^+_{\lambda_i}\right)^{n_{\lambda_i}} |0\rangle \tag{5.13}$$

In particular, the relation $[\hat{a}_{\lambda_i}, \hat{a}_{\lambda_j}]_\pm = 0$ is responsible for the symmetrization/antisymmetrization of the state $|\psi\rangle_\pm$ as expressed in (5.13), while $[\hat{a}_{\lambda_i}, \hat{a}^+_{\lambda_j}] = \delta_{ij}$ is responsible for the orthonormalization. Exemplarily

- $$a^+_\lambda a^+_\mu |0\rangle = \mathcal{N}\left(|\lambda\rangle \otimes |\mu\rangle \pm |\mu\rangle \otimes |\lambda\rangle\right)$$
$$\parallel$$
$$\pm a^+_\mu a^+_\lambda |0\rangle = \pm\mathcal{N}\left(|\mu\rangle \otimes |\lambda\rangle \pm |\lambda\rangle \otimes |\mu\rangle\right)$$

- $$\langle 0| a_\lambda a^+_\mu |0\rangle = \langle \lambda | \mu \rangle = \delta_{\lambda\mu}$$
$$\parallel$$
$$\langle 0| \delta_{\lambda\mu} \pm a^+_\mu a_\lambda |0\rangle = \delta_{\lambda\mu} \langle 0|0\rangle = \delta_{\lambda\mu}$$

We can now define $F_N^\pm$ as the Hilbert space spanned by all vectors as in (5.13) with $N = \sum_i n_{\lambda_i}$. The direct sum of all spaces with different particle number

$$F^\pm = \bigoplus_{N=0}^{\infty} F_N^\pm \tag{5.14}$$

is the Fock space of the system. Notice that the size and the form of the Hilbert spaces F_N depends on the nature of the particles. Moreover, while the algebra of $\{\hat{a}_{\lambda_i}, \hat{a}^+_{\lambda_i}\}$ is not closed in the Hilbert space $F_N^\pm$, it does in $F^\pm$.

$$\begin{tikzcd} \boxed{F_N^\pm} \arrow[r, bend right, "\hat{a}_{\lambda_i}"'] & \boxed{F_{N-1}^\pm} \arrow[l, bend right, "\hat{a}^+_{\lambda_i}"'] \end{tikzcd}$$