Quantum theory of condensed matter I

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Sheet 9

1. Occupation number representation

Consider a fermionic system with two single-particle states $|\alpha\rangle$ and $|\beta\rangle$ that span the (two-dimensional) *one*-particle Hilbert space. For instance, these can be spin-up and spin-down electron states in a single-level quantum dot or a hydrogen atom.

1. What are the dimensions of the *zero*-, *two*-, and *three*-particle Hilbert spaces? What is the dimension of the Fock space? Write down the basis of the Fock space explicitly as Slater determinants of the wave functions $\phi_\alpha(\mathbf{r}), \phi_\beta(\mathbf{r})$ and in the occupation number representation. **(2 Points)**
2. Calculate, in the Fock basis, the matrix representation of the creation and annihilation operators $\hat{c}_\mu, \hat{c}_\mu^\dagger$ ($\mu = \alpha, \beta$) and also of the occupation operators $\hat{n}_\mu = \hat{c}_\mu^\dagger \hat{c}_\mu$. **(2 Points)**
3. Using explicitly the matrix multiplication of the matrices calculated in 1.2, calculate the anticommutator relations
$$[\hat{c}_\mu, \hat{c}_\nu]_+ = \left[\hat{c}_\mu^\dagger, \hat{c}_\nu^\dagger\right]_+ = 0; \qquad \left[\hat{c}_\mu, \hat{c}_\nu^\dagger\right]_+ = \delta_{\mu\nu}$$
(2 Points)

2. Scalar potential as a 1-body operator

1. Consider a static scalar potential term $V(\mathbf{r})$ in a single-particle Hamiltonian and derive its many-body second-quantized form in the position and momentum representation. **(1 Point)**
2. Consider electron states in a deep rectangular quantum well of width a. The corresponding single-electron energy eigenvalues and eigenvectors are given by $\epsilon_{N\mathbf{k}} = \epsilon_N + \hbar^2 k^2/2m$ and $\psi_{N\mathbf{k}} = \sqrt{\frac{2}{aS}} \sin(N\frac{\pi}{a}z)\exp(i\mathbf{kr})$, $N = 1, 2, \cdots$ and $\mathbf{k}$ and $\mathbf{r}$ being two-dimensional vectors in the quantum well plane. Write down the second-quantized version of the many-body Hamiltonian in the above eigen basis when it is perturbed by an additional external potential $\delta V(t, \mathbf{r}, z)$. **(1 Point)**
3. For the particular cases of t-, $\mathbf{r}$-, or z-independent perturbations, describe qualitatively (in the framework of perturbation theory) which effects such perturbations may cause, based the presence/absense of certain couplings between the eigenstates of the unperturbed Hamiltonian. **(3 Points)**

3. Contact potential (oral)

Consider spin-1/2 fermions interacting only when their spatial separation is effectively zero, $V(\vec{r}_1 - \vec{r}_2) = V\delta(\vec{r}_1 - \vec{r}_2)$. Write down the associated two-particle operator in the second-quantized form, both in the position and momentum bases. Discuss distinctions between the short-range contact and long-range Coulomb interactions.

Frohes Schaffen!