Applications of Group Theory

Lectures	Tue	10:00 - 11:30	PHY 9.1.09
	Thu	10:00 - 11:30	PHY 9.1.09
Exercises	Fri	10:00 - 11:30	PHY 5.0.21

Sheet 4

---

1. Characters of the dihedral group D_n

Consider the generic proper group D_n which has a principal rotational axis C_n and n distinct dihedral axes C_2'.

1. Identify the conjugation classes of D_n. In particular, prove that the number of classes is $N_c = \frac{n+6}{2}$ for even n, while $N_c = \frac{n+3}{2}$ for odd n.
2. Prove that dihedral groups only admit irreducible representations of dimension 1 and 2. Prove, moreover:

$$\begin{aligned} n_1 = 4, \quad n_2 = \tfrac{n-2}{2}, \quad &\text{for even } n, \\ n_1 = 2, \quad n_2 = \tfrac{n-1}{2}, \quad &\text{for odd } n, \end{aligned}$$

   where n_i is the number of irreducible representation with dimension $i = 1, 2$.

3. Prove that, for every one dimensional representation it holds: $\chi(C_n) = \pm 1$ and $\chi(C_2') = \pm 1$. Conclude, by means of the orthogonality relation of the characters that, for the one dimensional representations it holds:

even n	C_n	C'_{2a}	C'_{2b}
A_1	1	1	1
A_2	1	-1	-1
B_1	-1	1	-1
B_2	-1	-1	1

odd n	C_n	C_2'
A_1	1	1
A_2	1	-1

4. Let $\omega := e^{2i\pi/n}$ and let $h \in \mathbb{Z}$. Consider the mappings $\rho^h : D_n \to GL_2(\mathbb{C})$ ($GL_2(\mathbb{C})$ is the group of invertible 2 x 2 complex matrices):

$$\rho^h(C_n^k) = \begin{pmatrix} \omega^{hk} & 0 \\ 0 & \omega^{-hk} \end{pmatrix}, \qquad \rho^h(C_n^k C_2') = \begin{pmatrix} 0 & \omega^{hk} \\ \omega^{-hk} & 0 \end{pmatrix}$$

   with $k = 1, 2, \dots n$.

   Prove that ρ^h for $h = 1, \dots, \frac{n-2}{2}$ or $\frac{n-1}{2}$ are 2 dimensional irreducible representations of D_n respectively for even and odd n. Calculate the corresponding character sets.

   Hint: Prove that ρ^h is a homomorphism, thus giving it the status of representation of D_n. Prove moreover that ρ^h is isomorphic to ρ^{n-h} and ρ^{n+h}, to restrict the range of h. Finally prove that ρ^0 and, for even n, $\rho^{n/2}$ are reducible representations.

Frohes Schaffen!