Quantentheorie II

Prof. Milena Grifoni Dr. Andrea Donarini

Blatt 10

1. Interacting fermions and bosons

Consider a system of interacting particles confined into a one dimensional harmonic potential. Let the interaction be local in space. The first quantization Hamiltonian has the form:

$$H = -\sum_{i=1}^{N} \frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + \frac{1}{2} m\omega^2 x_i^2 + \frac{U}{2} \sum_{i \neq j} \delta(x_i - x_j)$$

a) • Write the Hamiltonian in second quantization for a system of bosons of zero spin and of spin 1/2 fermions. In both cases use the position representation (field operators). **(2 Points)**

b) • Calculate for the bosonic and fermionic case the ground state energy of the two particle system to first order in the perturbation theory. **(2 Points)**

2. Lorentz transformations

Consider an inertial frame K' moving with constant speed v_1 along the x direction of a reference frame K and another inertial frame K'' moving with speed v_2 along the y direction of the reference frame K'.

a) • Calculate the matrix of the Lorentz transformation that transform the space-time coordinates of an event in the K frame into the ones of the same event in the K'' frame. Calculate also the inverse transformation. **(2 Points)**

b) Calculate the commutator between the single Lorentz transformations. *i.e.* The one between K and K' and the one between K' and K''.

c) What happens to the commutator in the case $v_1, v_2 \ll c$?

3. Klein-Gordon equation with a Coulomb potential

In presence of an external electrostatic potential of the form

$$V(r) = -\frac{e^2}{r}$$

the Klein-Gordon equation reads

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t} - V(r)\right)^2 + \hbar^2\Delta - m^2c^2\right]\psi(\vec{r},t) = 0.$$

a) Show that the stationary solutions of the Klein-Gordon equation have the form

$$\psi(\vec{r},t) = \frac{1}{r}\chi_\ell(r) Y_{\ell m}(\theta,\varphi) e^{-iEt/\hbar}$$

where $Y_{\ell m}(\theta,\varphi)$ are the spherical harmonics.

b) • Prove that the radial function $\chi_\ell(r)$ solves the equation

$$\frac{d^2}{dr^2}\chi_\ell(r) + \left(\frac{[E - V(r)]^2 - E_0^2}{\hbar^2 c^2} - \frac{\lambda}{r^2}\right)\chi_\ell(r) = 0$$

and give the explicit form of the constants E_0 and λ. **(3 Points)**

c) • Calculate the discrete energy spectrum of the bound states. You can resort to the well known derivation of the energy spectrum of the non-relativistic hydrogen atom. Which eigenenergies do you obtain in the non relativistic limit? **(3 Points)**

Frohes Schaffen!

Return the solution of the exercises marked with • by Thursday the 8th of January at 10:00.