Quantentheorie II

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Blatt 12

1. Scattering in the Born approximation

• Calculate the scattering amplitude and also the total scattering cross-section for the scattering of a particle with mass m and wavevector $\vec{k}$ due to a potential

$$V(\vec{r}) = V_0 \exp(-r/a)$$

using the Born approximation. **(2 Points)**

2. Occupation number representation

Let us consider a bosonic system with two single particle states $|\phi_1\rangle$ and $|\phi_2\rangle$ (for simplicity we neglect the spin degree of freedom) that span the (two-dimensional) *one*-particle Hilbert space.

a) • Which dimension has the *two*-particle Hilbert space? Which dimension has the Fock space? Write down the form of the basis of the Fock space explicitly as products of the wave functions $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$ and in the occupation number representation. **(2 Points)**

b) • Calculate, in the Fock basis, the matrix representation of the creation and annihilation operators c_i, $c_i^\dagger$ $(i = 1, 2)$ and also of the occupation operators $n_i = c_i^\dagger c_i$. **(3 Points)**

c) Verify the commutator relations

$$[c_i, c_j] = [c_i^\dagger, c_j^\dagger] = 0\,, \quad [c_i, c_j^\dagger] = \delta_{ij}$$

explicitly using matrix multiplication of the matrices calculated at point b).

d) • Consider a Hamilton operator

$$\hat{H} = \hat{T} + \hat{V}\,,$$

where $\hat{T}$ is a single particle operator and $\hat{V}$ a two particle one. With respect to the single particle basis $|\phi_i\rangle$ the matrix elements are:

$$\langle\phi_i|\,\hat{T}\,|\phi_i\rangle = \epsilon\,, \quad \langle\phi_i|\,\hat{T}\,|\phi_j\rangle = t \text{ for } i \neq j$$
$$\langle\phi_1,\phi_2|\,\hat{V}\,|\phi_1,\phi_2\rangle = U_{12}\,, \quad \langle\phi_i,\phi_i|\,\hat{V}\,|\phi_i,\phi_i\rangle = U$$
$$\langle\phi_1,\phi_2|\,\hat{V}\,|\phi_2,\phi_1\rangle = J\,, \quad \langle\phi_i,\phi_1|\,\hat{V}\,|\phi_i,\phi_2\rangle = \tilde{t}$$
$$\langle\phi_1,\phi_1|\,\hat{V}\,|\phi_2,\phi_2\rangle = \langle\phi_2,\phi_2|\,\hat{V}\,|\phi_1,\phi_1\rangle = t_2$$

where the notation is such that, *e.g.*:

$$\langle\phi_1,\phi_2|\,\hat{V}\,|\phi_2,\phi_1\rangle \equiv \int \mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\phi_1^*(\mathbf{r}_1)\phi_2^*(\mathbf{r}_2)V(\mathbf{r}_1,\mathbf{r}_2)\phi_1(\mathbf{r}_2)\phi_2(\mathbf{r}_1)\,.$$

Remember that, in second quantization, a single and two particle operators are respectively written as:

$$\hat{T} = \sum_{\lambda,\mu} c_\lambda^\dagger\langle\phi_\lambda|\,\hat{T}\,|\phi_\mu\rangle c_\mu\,, \quad \hat{V} = \frac{1}{2}\sum_{\lambda\mu\lambda'\mu'} c_\lambda^\dagger c_\mu^\dagger\langle\phi_\lambda,\phi_\mu|\,\hat{V}\,|\phi_{\lambda'},\phi_{\mu'}\rangle c_{\mu'}c_{\lambda'}\,,$$

where $|\phi_\lambda\rangle$ represent a generic single particle basis and $c^\dagger_\lambda$ the corresponding creation operator. Write the operator $\hat{H}$ in second quantization and in the matrix representation (starting from the single particle basis introduced). Calculate the eigenvalues and eigenvectors for $\hat{H}$.

(3 Points)

e) Again, write $\hat{H}$ in second quantization, but this time as a single particle basis use the eigenvectors of $\hat{T}$. Which is the connection between this creation and annihilation operators and the ones considered in the points a)-d)? Is this a unitary transformation?

Frohes Schaffen!

Return the solution of the exercises marked with • by Monday 26th of January at 10:00.