What can we learn from Drude

$$m\dot{\vec{v}} = \underbrace{-e\vec{E}}_{\text{electric force}} - \underbrace{\frac{m\vec{v}}{\tau_m}}_{\text{friction}} \qquad (*)$$

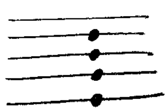
No Fermi statistics and no Bloch theorem. We are considering particles that collide on "something" that reduces their velocity. This result is already in contrast with quantum mechanics since it cannot happen that all electrons have the same velocity. The solution will come through $\langle \vec{v} \rangle = 0$. Stationary solution of $(*)$

$$\vec{v} = + \frac{e\vec{E}\tau_m}{m}$$

$$\vec{J} = +en\vec{v} = \frac{e^2 n \tau_m}{m}\vec{E} = \sigma \vec{E}$$

$$\boxed{\sigma = \frac{e^2 n \tau_m}{m}}$$ Drude formula (1900)

a) How does the Fermi statistics cope with this formula?



acceleration implies a change in energy that only the electrons at the Fermi level can accept

$$n \overset{3D}{=} \frac{1}{3\pi^2} k_F^3$$

$$\sigma = \frac{e^2}{3\pi^2}\frac{k_F^3 \tau_m}{m} = \frac{e^2}{12\pi^3\hbar} 4\pi k_F^2 \frac{\hbar k_F}{m}\tau_m = \frac{e^2}{12\pi^3\hbar} S_F \cdot \overbrace{v_F \cdot \tau_m}^{\ell_m}$$

(Einstein formula)

b) $$\sigma = \frac{e^2}{3\pi^2}\frac{k_F^3\tau_m}{m} = \frac{e^2}{3\pi^2}\underbrace{\frac{\hbar^2 k_F^2}{m^2}}_{v_F^2}\frac{m}{\hbar^2}k_F\tau_m$$

Which is the density of states at the Fermi level?

$$\rho(\varepsilon_F) = \frac{2}{V}\sum_k \delta(\varepsilon_k - \varepsilon_F) = \cancel{2}\frac{1}{(\cancel{2}\pi)^{\cancel{3}}}\,2\int \cancel{4\pi} k^{\cancel{2}}\,dk\,\frac{m}{\hbar^2 \cancel{k}}\,\delta(k-k_F)$$

$$= \frac{1}{\pi^2}\frac{k_F m}{\hbar^2}$$

$$\sigma = e^2\frac{v_F^2\tau_m}{3}\rho(\varepsilon_F) = e^2\rho(\varepsilon_F)\,D$$ ⟿ diffusion coefficient

↳ density of states at the Fermi energy.

Yet another reformulation to prove that Einstein relation is valid in generic dimensions.

$$n = \frac{N}{L^d}$$ d is the dimension of the conductor

$$\sigma = \frac{e^2 n\tau_m}{m} = \frac{e^2}{m}\frac{N}{L^d}\tau_m = \frac{e^2}{\hbar}\frac{N}{\frac{1}{2}mv_F^2}\underbrace{\boxed{\frac{1}{d}v_F^2\tau_m}}_{D}\cdot\frac{\hbar}{L^d}\cdot\frac{d}{2}$$

$$\frac{\varepsilon_F}{N}$$

$$N = 2\sum_k \theta(\varepsilon_F - \varepsilon_k) = 2L^d\frac{\Omega_d}{(2\pi)^d}\int_0^{k_F} dk\,k^{d-1} = 2L^d\frac{\Omega_d}{(2\pi)^d}\frac{k_F^d}{d}$$

$\Omega_d = 1, 2\pi, 4\pi$ 1,2,3 dimensions

$$\frac{\varepsilon_F}{N} = \frac{\hbar^2 k_F^2}{2m}\cdot\frac{d\,(2\pi)^d}{2L^d k_F^d\Omega_d} = \frac{\hbar^2}{2m}\frac{(2\pi)^d}{\Omega_d}\frac{d}{2L^d}\frac{1}{k_F^{d-2}} = \frac{d}{2L^d}\frac{1}{\rho(\varepsilon_F)}$$

$$\rho(\varepsilon_F) = \frac{2}{L^d}\sum_k \delta(\varepsilon_k - \varepsilon_F) = \frac{2}{(2\pi)^d}\Omega_d\int_0^\infty dk\,k^{d-1}\frac{m}{\hbar^2 k}\delta(k-k_F) = 2\frac{\Omega_d}{(2\pi)^d}\frac{m}{\hbar^2}k_F^{d-2}$$

$$\sigma = \frac{e^2}{\hbar}\rho(\varepsilon_F)\frac{2L^{\cancel{d}}}{\cancel{d}}\cdot D\,\cancel{\frac{\hbar}{L^d}}\frac{\cancel{d}}{2} = \frac{e^2}{\hbar}\rho(\varepsilon_F)\,D$$

c) The Thouless time and energy:

$$\sigma = \frac{ne^2\tau_m}{m}$$

$$= \frac{e^2}{\hbar}\frac{v_F^2\tau_m}{d}\frac{N}{L^d}\frac{d}{2}\frac{\hbar}{\varepsilon_F} = \frac{e^2}{\hbar}\frac{D\hbar}{L^2}\frac{N}{2\varepsilon_F}\cdot d\,\frac{1}{L^{d-2}}$$

$\frac{2\varepsilon_F}{N} = \Delta$ mean level spacing.

$\frac{D\hbar}{L^2} = E_T \leftarrow$ Thouless energy

$g = \frac{E_T}{\Delta}$ is also called dimensionless conductance.

N.B. in 2 dimension $G = \sigma$ $\left(G = \sigma L^{d-2} \right)$

$$G = \frac{e^2}{\hbar}\frac{E_T}{\Delta}d$$

Boltzmann transport formalism

1 - Non-equilibrium distribution and current: Let's assume for the moment to have a spatially uniform gas of electrons. In equilibrium we will have that this gas is distributed according to the relation $f^\circ(\vec{k})$ where f° is the Fermi function (better saying $f^\circ(\varepsilon_{\vec{k}})$). Which is then the current transported by this system of electrons?

$$\vec{J} = en\vec{v} \qquad n = \frac{1}{V}, \quad e = \text{the electronic charge}$$

$$\vec{v}\ ?$$

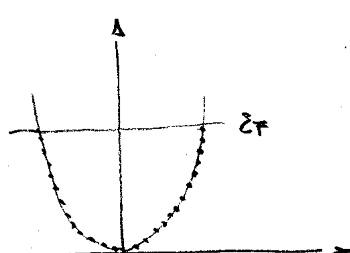
Since we are speaking of the total current $\vec{v} = \sum_{\vec{k}} \vec{v}_k f^\circ(\varepsilon_{\vec{k}})$

To be more precise we should speak about current density:

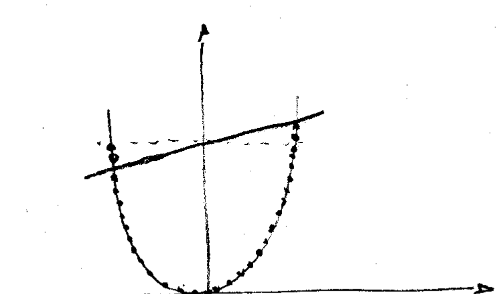
$$\vec{v}_k = \frac{1}{\hbar}\frac{\partial E}{\partial \vec{k}} \overset{\text{free el}}{=} \frac{\hbar\vec{k}}{m},$$

Since $\frac{\partial \varepsilon_k}{\partial \vec{k}}$ is an odd function of $\vec{k}$ while $f^\circ(\varepsilon_{\vec{k}})$ is even, as the current vanishes.

What happens if an electric field is applied?



$\vec{E} = 0$
$\vec{J} = 0$

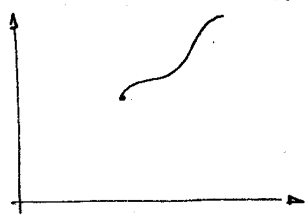


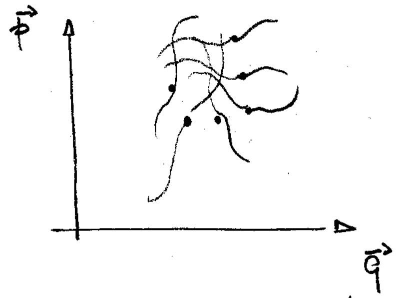
$\vec{E} \neq 0$
$\vec{J} \neq 0$

2 – Boltzmann equation:

We would like to find: i) a semiclassical distribution able to describe the motion of N electrons ($N \sim 10^{23}$). ii) the dynamics of this distribution function.

Dynamics of N particles:





Let's assume that $H = \sum_{i=1}^{N} H_i$ that is: we are dealing with a set of independent particles.

$$G(q,p,t) = \sum_{i=1}^{N} \delta(q - q_i(t))\,\delta(p - p_i(t))$$ is the "simplest" classical distribution.

$$\frac{\partial G}{\partial t} = \sum_{i=1}^{N} \left[\frac{\partial}{\partial q} \delta(q - q_i(t))\,(-1)\dot{q}_i\,\delta(p - p_i(t)) + \frac{\partial}{\partial p} \delta(p - p_i(t))(-\dot{p}_i)\,\delta(q - q_i) \right] =$$

$$= -\sum_{i=1}^{N} \left[\frac{\partial}{\partial q} \delta(q - q_i(t))\,\delta(p - p_i(t))\,\frac{p_i}{m} + \frac{\partial}{\partial p} \delta(p - p_i)\left(-\frac{\partial V}{\partial q_i}\right)\delta(q - q_i) \right]$$

$$= -\frac{p}{m}\frac{\partial}{\partial q} G + \frac{dV}{dq}\frac{\partial}{\partial p} G = -\frac{p}{m}\frac{\partial}{\partial q} G - F\frac{\partial}{\partial p} G .$$

The only problem with G is that we would like to deal with a much better behaved distribution f that is able to capture some macroscopic property of the system. Nevertheless a similar equation to that of G will be satisfied also by f. Notice that indeed

$$N = \frac{1}{V}\int \frac{d\vec{q}\,d\vec{p}}{\hbar^3}\,\frac{V}{(2\pi)^3}\, f(\vec{q},\vec{p},t) = \int dN$$

- The number of particles is also conserved if we let evolve the coordinates with the same evolution of the N points:

$$f(\vec{q},\vec{p},t)\,d\vec{q}\,d\vec{p} = f(\vec{q}',\vec{p}',t')\,d\vec{q}'\,d\vec{p}' = dN$$

- The phase space volume is conserved under coordinate transformation that follow an Hamilton dynamics.

Let's infact consider the infinitesimal transformation

$$\vec{q}\,' = \vec{q} + \dot{\vec{q}}\,dt$$

$$\vec{p}\,' = \vec{p} + \dot{\vec{p}}\,dt$$

$$d\vec{q}\,'d\vec{p}\,' = \underbrace{\begin{vmatrix} \frac{\partial q_i'}{\partial q_j} & \frac{\partial p_i'}{\partial q_j} \\ \frac{\partial q_i'}{\partial p_j} & \frac{\partial p_i'}{\partial p_j} \end{vmatrix}}_{} d\vec{q}\,d\vec{p} \quad \Rightarrow \quad J = \begin{vmatrix} \delta_{ij} + \frac{\partial^2 H}{\partial q_j \partial p_i} dt & -\frac{\partial^2 H}{\partial q_j \partial q_i} dt \\ -\frac{\partial^2 H}{\partial p_i \partial p_j} dt & \delta_{ij} - \frac{\partial^2 H}{\partial p_j \partial q_i} dt \end{vmatrix}$$

$$= 1 + \sum_i \left(\cancel{\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i}} \right) dt + o(dt) = 1 + o(dt)$$

A way to prove it is that for an Hamiltonian dynamics, putting together $\vec{q}$ and $\vec{p}$ $\quad \dot{x} = u(x)$

$$\frac{1}{\delta V} \frac{d}{dt} \delta V = \operatorname{div} u \equiv \sum_i \frac{\partial u_i}{\partial x_i}$$

proof: let's assume $\quad u(x_1, \dots, x_n) = (u_1(x_1), 0, 0, 0, \dots, 0)$

$$(x_1, x_1 + dx_1) \times (x_2, x_2 + dx_2) \times \dots \times (x_n, x_n + dx_n)$$

$$\Rightarrow \quad \delta V = \prod_h dx_h$$

$$\delta V(t+dt) = \left(x_1 + u_1(x_1)dt,\ x_1 + dx_1 + u_1(x_1 + dx_1)dt \right) \times \dots \times (x_n, x_n + dx_n)$$

$$= \left(dx_1 + \frac{\partial u_1}{\partial x_1} dx_1 dt \right) \times \dots \times$$

$$\delta V(t+dt) - \delta V(t) = \frac{\partial u_1}{\partial x_1} dx_1 dx_2 \cdots dx_n dt \Rightarrow \frac{d\,\delta V}{dt} = \frac{\partial u_1}{\partial x_1} \delta V$$

$\Rightarrow$ since we only want the linear terms in dt we obtain the result.

But for the Hamiltonian dynamics $\quad \operatorname{div} u = 0 \quad$ since

$$\operatorname{div} u = \sum_i \left(\frac{\partial}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial q_i} \right) = 0$$

From the 2 observations it follows that

$$f(\vec{q}\,',\vec{p}\,',t') = f(\vec{q},\vec{p},t) \qquad \forall t'$$

$\Rightarrow$ the total derivative with respect of time of the function $f(q(t),p(t),t)$ vanishes.

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i =$$

$$= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} \right) \qquad \Rightarrow \quad \frac{\partial f}{\partial t} = \{H, f\}$$

For $H = \sum_i \frac{p_i^2}{2m} + V(\vec{q})$

$$\boxed{\frac{\partial f}{\partial t} = -\frac{\vec{p}}{m}\cdot\nabla_{\vec{q}} f - \vec{F}\cdot\nabla_{\vec{p}} f}$$

All what is not single particle or we want to treat as a perturbation is put into the collisional term

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{coll}$$

Examples

- electrons scattered among themselfes
- electrons scattered on impurities
- electrons scattered on phonons

Summarizing

$$\partial_t f + \partial_{\vec{q}} f \cdot \frac{\vec{p}}{m} + \partial_{\vec{p}} f \cdot \left(e\vec{E} + e\frac{\vec{p}}{m}\times\vec{B} \right) = [\partial_t f]_{coll}$$

$$f = f(\vec{q},\vec{p},t)$$

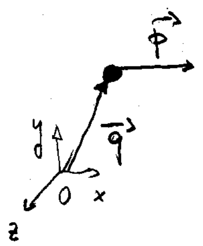
$$N = \frac{1}{V}\int d\vec{q}\; \frac{V}{(2\pi)^3\hbar^3}\int d^3\vec{p}\; f(\vec{q},\vec{p},t) \qquad \leftarrow \text{normalization}$$

$$f_{\vec{p}}(t) = \frac{1}{V}\int d^3\vec{q}\; f(\vec{q},\vec{p},t) \qquad \leftarrow \text{momentum distribution}$$

Impurity scattering: As an example of collision contribution to the Boltzmann equation we consider the scattering due to static impurities.
It is remarkable how in the Drude idea this scattering was thought to be against ions. After Bloch we know that in a perfect crystal electrons would not suffer of any scattering. Impurity is thus: ① an imperfection in the crystal structure ② the donor (acceptor) atoms in the doped semiconductor.

For us:

electron - classical



q.m

$\hbar\vec{k} = \vec{p} = \frac{m}{\hbar}\frac{\partial E}{\partial \vec{k}}$.

$\dot{\vec{q}}$

The impurity is represented by a potential:

$V^{imp}(\vec{q}) = \sum_i V(\vec{q}-\vec{R}_i)$ where $\vec{R}_i$ is the location of the i-th impur.

N.B. This kind of scattering could in principle be inserted exactly into the Boltzmann equation, being a single particle term. Nevertheless, the complexity of the exact approach justifies the following approximation.

The rate of scattering $k \to k'$ can be calculated in Fermi golden rule.

$$W_{kk'} = W(k\to k') = \frac{2\pi}{\hbar}\left|\langle \vec{k}'|\hat{V}^{imp}|\vec{k}\rangle\right|^2 \delta(\varepsilon_k - \varepsilon_{k'})$$

$$\langle \vec{k}'|\hat{V}_{imp}|\vec{k}\rangle = \int d\vec{q}\,\frac{1}{V}\,e^{-i(\vec{k}'-\vec{k})\vec{q}}\,V^{imp}(\vec{q}) =$$

$$= \int d\vec{q}\,\frac{1}{V}\,e^{-i(\vec{k}'-\vec{k})\vec{q}}\sum_i V(\vec{q}-\vec{R}_i)$$

$$= \int d\vec{q}\,\frac{1}{V}\,e^{-i(\vec{k}'-\vec{k})\vec{q}}\sum_i V(\vec{q})\,e^{-i(\vec{k}-\vec{k}')\cdot\vec{R}_i}$$

$$= \sum_i \tilde{V}(\vec{k}-\vec{k}')\,e^{-i(\vec{k}-\vec{k}')\cdot\vec{R}_i}$$

$$|\langle \vec{k}'|\hat{V}^{imp}|\vec{k}\rangle|^2 = |\tilde{V}(\vec{k}-\vec{k}')|^2 \left|\sum_j e^{+i(\vec{k}-\vec{k}')\cdot\vec{R}_j}\right|^2$$

$$\overset{\tilde{\vec{k}}=\vec{k}-\vec{k}'}{=} |\tilde{V}(\tilde{\vec{k}})|^2 \sum_{ij} e^{i\tilde{\vec{k}}\cdot(\vec{R}_i-\vec{R}_j)}$$

$$= |\tilde{V}(\tilde{\vec{k}})|^2 \left[N_{imp} + \sum_{i\neq j} e^{-i\tilde{\vec{k}}\cdot(\vec{R}_i-\vec{R}_j)} \right]$$

Since the smallest possible $|\tilde{\vec{k}}| = \frac{2\pi}{L}$ and R_i, R_j are uniformly distributed over L^3, the second term is negligible with respect to N_{imp}.

The collision term of the Boltzmann equation looks like:

$$\left(\frac{\partial f_{\vec{k}}}{\partial t}\right)_{coll.} = \sum_{k'} W_{kk'}\Big[\underbrace{f_{k'}(1-f_k)}_{k'\to k} - \underbrace{f_k(1-f_{k'})}_{k\to k'}\Big] = \sum_{k'} W_{kk'}[f_{k'}-f_k]$$

<u>Relaxation time approximation</u> Now let's consider the BE in presence of an homogeneous, constant electric field $\vec{E}$ $(\vec{B}=0)$. We look for an homogeneous steady state solution of the Boltzmann equation.

$$-e\vec{E}\cdot\frac{1}{\hbar}\partial_{\vec{k}} f_{\vec{k}} = \sum_{\vec{k}'} W_{kk'}(f_{k'}-f_k)$$

N.B. for homogeneous distributions $f \equiv f_{\vec{k}}$.

- elastic impurity scattering
- spherically symmetrical impurity potential
- weak electric field $\quad eE\,L_m \ll E_F$
- free electrons $\left(\varepsilon_{\vec{k}} = \frac{1}{2m}\hbar^2\vec{k}^2\right)$

$$W_{\vec{k}\vec{k}'} = W(\varepsilon_k, \hat{k}\cdot\hat{k}')\,\delta(\varepsilon_k-\varepsilon_{k'}) \qquad \text{(Exercises 1.)}$$

Since $\vec{E}$ is meant to be a perturbation, we write

$$f_{\vec{k}} = f_{\vec{k}}^{(0)} + f_{\vec{k}}^{(1)} + f_{\vec{k}}^{(2)} + \dots \quad \text{with } f_{\vec{k}}^{(0)} = f_0(\varepsilon_k) \text{ Fermi distribution.}$$

this result follows within the Boltzmann approach due to H-theorem + Fermi-Dirac statistics.

The left hand side of the BE to first order in $\vec{E}$:

$$-e\vec{E}\cdot\frac{1}{\hbar}\frac{\partial f^{0}}{\partial\vec{k}} = -e\vec{E}\cdot\frac{1}{\hbar}\frac{\partial f_0}{\partial\varepsilon_k}\underbrace{\frac{\partial\varepsilon_k}{\partial\vec{k}}}_{\frac{\hbar^2}{m}\vec{k}=\hbar\vec{v}_k} = -\frac{\hbar e}{m}\vec{E}\cdot\vec{k}\,\frac{\partial f_0}{\partial\varepsilon_k}$$

The right hand side

$$\frac{V}{(2\pi)^3}\int d\vec{k}'\; W(\vec{k},\vec{k}')\left(f_{\vec{k}'}^{(1)} - f_{\vec{k}}^{(1)}\right) = I_1 + I_2$$

$$I_2 = -f_{\vec{k}}^{(1)}\frac{V}{(2\pi)^3}\int d\Omega_{k'}\int dk'\,k'^2\; W(\varepsilon_{k'},\hat{k}\cdot\hat{k}')\;\delta(\varepsilon_k-\varepsilon_{k'}) \qquad \frac{\hbar^2 k'^2}{2m}=\varepsilon_{k'} \quad k'=\frac{\sqrt{2m\varepsilon_{k'}}}{\hbar}$$

$$d\varepsilon_{k'} = \frac{\hbar^2}{m}k'dk'$$

$$= -f_{\vec{k}}^{(1)}\cdot\frac{V}{(2\pi)^3}\int_0^{2\pi}d\varphi_{k'}\int_{-1}^{1}d\cos\theta_{k'}\int_0^{\infty}d\varepsilon_{k'}\,\sqrt{\varepsilon_{k'}}\cdot\frac{\sqrt{2}\,m^{3/2}}{\hbar^3}W(\varepsilon_{k'},\cos\theta_{k'})\,\delta(\varepsilon_k-\varepsilon_{k'}) \qquad dk'=\sqrt{\frac{m}{2\hbar^2}}\frac{1}{\sqrt{\varepsilon_{k'}}}d\varepsilon_{k'}$$

$$= -f_{\vec{k}}^{(1)}\frac{V}{(2\pi\hbar)^3}\cdot 2\pi\sqrt{2}\,m^{3/2}\sqrt{\varepsilon_k}\int_{-1}^{1}d\cos\theta\; W(\varepsilon_k,\cos\theta)$$

$$= -f_{\vec{k}}^{(1)}\cdot\frac{1}{\tau(\varepsilon_k)} \qquad \tau^{-1}(\varepsilon_k) = N_0(\varepsilon_k)\int\frac{d\hat{k}'}{4\pi}\,W(\hat{k}\cdot\hat{k}',\varepsilon_k)$$

$$N_0(\varepsilon_k) = \frac{V\sqrt{2m^3\varepsilon_k}}{2\pi^2\hbar^3} \quad \text{\# of states per unit energy}$$

$$W(\hat{k}\cdot\hat{k}',\varepsilon_k) = \frac{2\pi}{\hbar}N_{imp}\left|\tilde{V}(\vec{k}-\vec{k}')\right|^2$$

Again I_2

$$I_2 = -f^{(1)}_{\vec{k}} \sum_{\vec{k}'} W(\vec{k},\vec{k}') =$$

$$= -f^{(1)}_{\vec{k}} \int \frac{d\hat{k}'}{4\pi} \int d\varepsilon_{k'} V\rho(\varepsilon_{k'})\, W(\varepsilon_k, \hat{k}\cdot\hat{k}')\, \delta(\varepsilon_k - \varepsilon_{k'})$$

$$= -f^{(1)}_{\vec{k}}\, V\rho(\varepsilon_k) \int \frac{d\hat{k}'}{4\pi} W(\varepsilon_k, \hat{k}\cdot\hat{k}')$$

very general, one needs only the isotropic hypothesis.

$$I_1 = V\rho(\varepsilon_k) \int \frac{d\hat{k}'}{4\pi} W(\varepsilon_k, \hat{k}\cdot\hat{k}')\, f^{(1)}_{\vec{k}'}$$

Putting all together:

$$0 = -\frac{\hbar e}{m} \vec{E}\cdot\vec{k}\, \frac{\partial_0(\varepsilon_k)}{\partial \varepsilon_k} + V\rho(\varepsilon_k)\left[\int \frac{d\hat{k}'}{4\pi} W(\hat{k}\cdot\hat{k}', \varepsilon_k)\left(f^{(1)}_{\vec{k}} - f^{(1)}_{\vec{k}'}\right)\right] \quad *$$

One can use an expansion in terms of the Legendre polynomials:

$$W(\hat{k}\cdot\hat{k}', \varepsilon_k) = \sum_{l=0}^{\infty} W_l(\varepsilon_k)\, P_l(\hat{k}\cdot\hat{k}') \qquad P_0(x) = 1 \quad (A)$$
$$P_1(x) = x$$

$$W_l(\varepsilon_k) = (2l+1)\int \frac{d\hat{k}'}{4\pi} P_l(\hat{k}\cdot\hat{k}')\, W(\hat{k}\cdot\hat{k}', \varepsilon_k) \quad (B)$$

The addition theorem says:

$$\int \frac{d\hat{k}'}{4\pi} P_l(\hat{k}\cdot\hat{k}')\, P_{l'}(\hat{k}'\cdot\hat{E}) = \frac{1}{2l+1}\,\delta_{l,l'}\, P_l(\hat{k}\cdot\hat{E}) \quad (C)$$

$$f^{(1)}_{\vec{k}} = \sum_{l=0}^{\infty} f^{(1)}_l(\varepsilon_k)\, P_l(\hat{k}\cdot\hat{E})$$

$$(*) \Rightarrow \quad 0 = -\frac{\hbar e}{m} P_1(\hat{k}\cdot\hat{E})\, E k\, \frac{\partial f_0}{\partial \varepsilon_k} + V\rho(\varepsilon_k)\left[W_0(\varepsilon_k)\sum_{l=0}^{\infty} f^{(1)}_l(\varepsilon_k)\, P_l(\hat{k}\cdot\hat{E}) \right.$$
$$\left. - \int \frac{d\hat{k}'}{4\pi} \sum_{l=0}^{\infty} W_l(\varepsilon_k)\, P_l(\hat{k}\cdot\hat{k}') \sum_{l'=0}^{\infty} f^{(1)}_{l'}(\varepsilon_k)\, P_{l'}(\hat{k}'\cdot\hat{E})\right]$$

$$0 = -\frac{\hbar e}{m} P_1(\hat{k}\cdot\hat{E})\, E k \frac{\partial f_0}{\partial \varepsilon_k} +$$

$$+ V\rho(\varepsilon_k) \sum_{l=0}^{\infty} \left[W_0(\varepsilon_k) f_l^{(1)}(\varepsilon_k) - \frac{W_l(\varepsilon_k) f_l^{(1)}(\varepsilon_k)}{2l+1} \right] P_l(\hat{k}\cdot\hat{E}) =$$

$$= \sum_{l=0}^{\infty} \left\{ -\frac{\hbar e}{m} \delta_{l1} E k \frac{\partial f_0}{\partial \varepsilon_k} + V\rho(\varepsilon_k) \left[W_0(\varepsilon_k) f_l^{(1)}(\varepsilon_k) - \frac{W_l(\varepsilon_k) f_l^{(1)}(\varepsilon_k)}{2l+1} \right] \right\} P_l(\hat{k}\cdot\hat{E})$$

All coefficients must be equal to 0

$l=0$ $\qquad W_0(\varepsilon_k) f_0^{(1)}(\varepsilon_k) - W_0(\varepsilon_k) f_0^{(1)}(\varepsilon_k) = 0$ $\qquad$ trivial

$l=1$ $\qquad + \frac{\hbar e}{m} E k \frac{\partial f_0}{\partial \varepsilon_k} = V\rho(\varepsilon_k) \left[W_0(\varepsilon_k) - \frac{W_1(\varepsilon_k)}{3} \right] f_1^{(1)}(\varepsilon_k)$

$f_l^{(1)}(\varepsilon_k) = 0$ for $l \neq 1$ this is the first result.

$$f_{l=1}^{(1)}(\varepsilon_k) = \left(W_0 - \frac{W_1}{3} \right)^{-1} \frac{\hbar e}{m} E k \frac{\partial f_0}{\partial \varepsilon_k}$$

$$\Rightarrow \quad f_{\vec{k}}^{(1)} = \frac{\hbar e}{m} \vec{E}\cdot\vec{k}\; \tau_m(\varepsilon_k) \frac{\partial f_0}{\partial \varepsilon_k}$$

where the momentum relaxation time $\tau_m(\varepsilon_k)$

$$\frac{1}{\tau_m} := V\rho(\varepsilon_k) \left(W_0 - \frac{W_1}{3} \right) =$$

$$= V\rho(\varepsilon_k) \int \frac{d\hat{k}'}{4\pi} W(\hat{k}\cdot\hat{k}', \varepsilon_k) \left(1 - \cancel{3} \frac{\hat{k}\cdot\hat{k}'}{\cancel{3}} \right)$$

$$= \int \frac{d^3k'}{4\pi} W(\vec{k}, \vec{k}') \left(1 - \cos\theta_{kk'} \right)$$

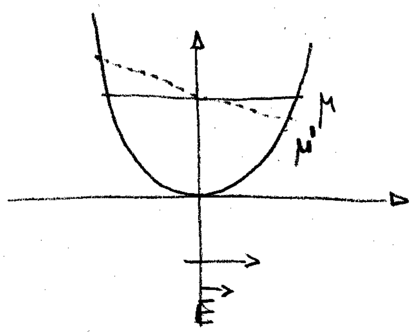
notice that the momentum relaxation time includes the angle.

The solution to the BE is thus, to first order in $\vec{E}$

$$f = f_0(\varepsilon_k) + \frac{\hbar e}{m} \tau_m(\varepsilon_k) \frac{\partial f_0}{\partial \varepsilon_k} \vec{E}\cdot\vec{k}$$

$$\simeq f_0\left(\varepsilon_k + e\vec{E}\cdot\frac{\hbar\vec{k}}{m}\tau_m(\varepsilon_k)\right) =$$

$$= f\left(\underbrace{\varepsilon_k + e\vec{E}\cdot\frac{\hbar\vec{k}}{m}\tau_m(\varepsilon_k) - \mu}_{-\mu'(\vec{k})}\right)$$



$$e\vec{E}\cdot\vec{v}_F\cdot\tau_m = eE\ell_m = U\frac{L_m}{L}$$

$$\left.\begin{aligned} L &= 10^{-2}\,m \\ L_m &= 10^{-8}\,m \\ U &= 1V \end{aligned}\right\} \quad \max(\mu'-\mu) \approx 10^{-6}\,eV \ll E_F$$

Classical conductivity

$$f_{\vec{k}} = f_0(\varepsilon_k) + e\,\tau_m(\varepsilon_k)\frac{\partial f_0}{\partial \varepsilon_k}\frac{\hbar}{m}\vec{k}\cdot\vec{E}$$

$$\vec{J} = 2\,\frac{-e}{\cancel{V}}\cancel{V}\int\frac{d\vec{k}}{(2\pi)^3}(-e)\,\tau_m(\varepsilon_k)\frac{\hbar\vec{k}}{m}\left(-\frac{\partial f_0}{\partial\varepsilon_k}\right)\frac{\hbar\vec{k}}{m}\cdot\vec{E}$$

$$= 2e^2\int\frac{d\vec{k}}{(2\pi)^3}\,\tau_m(\varepsilon_k)\,\vec{v}\left(-\frac{\partial f_0}{\partial\varepsilon_k}\right)\vec{v}\cdot\vec{E}$$

$$\Rightarrow \quad \sigma_{\alpha\beta} = 2e^2\int\frac{d\vec{k}}{(2\pi)^3}\,v_{\vec{k},\alpha}v_{\vec{k},\beta}\,\tau_m(\varepsilon_k)\left(-\frac{\partial f_0}{\partial\varepsilon_k}\right)$$

$T \ll \varepsilon_k$
isotropic system

$$= 2\frac{e^2}{(2\pi)^3}\frac{\hbar^2}{m^2}\int d\hat{k}\int dk\,k^2\,k^2\,\tau_m(\varepsilon_k)\,\delta(\varepsilon_k-\varepsilon_F)$$

$$= 2\frac{e^2}{(2\pi)^3}\frac{1}{m}4\pi\,k_F^3\,\tau_m(\varepsilon_k) = \frac{e^2 n\tau_m}{m}.$$