Mesoscopic Physics

Dr. Andrea Donarini
Dr. Miriam del Valle

Room 5.01.01
Wednesdays at 15:15

Sheet 2

1. Naïve distribution function

Consider the trial distribution function given by $f(q,p) = \mathrm{Tr}\{\hat{\rho}\,\delta(q-\hat{q})\;\delta(p-\hat{p})\}$. Prove that this function is complex and thus represents a bad choice.
Hint: It is enough to provide a counter example. Consider for instance the pure state density operator defined by $\hat{\rho} = |\psi\rangle\langle\psi|$, built from a wave function such as $\psi(x) = \sqrt{\alpha/\pi}\sin(\frac{\alpha x}{\hbar})$, with x defined in the interval $[0, 2\pi\hbar/\alpha]$.

2. Wigner distribution

• The Wigner distribution function is defined as

$$f_{\mathrm{W}}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}y\;\left\langle q-\frac{y}{2}\right|\hat{\rho}\left|q+\frac{y}{2}\right\rangle e^{\mathrm{i}\frac{py}{\hbar}}$$

Prove that it has the following properties:

(i) It is real. That is, $\mathrm{Im}(f_{\mathrm{W}}) = \frac{1}{2}(f_{\mathrm{W}} - f_{\mathrm{W}}^*) = 0$.

(ii) f_{W} satisfies:

$$\begin{aligned}
\int \mathrm{d}p\, f_{\mathrm{W}}(q,p) &= \langle q|\hat{\rho}|q\rangle \\
\int \mathrm{d}q\, f_{\mathrm{W}}(q,p) &= \langle p|\hat{\rho}|p\rangle \\
\int \mathrm{d}q\,\mathrm{d}p f_{\mathrm{W}}(q,p) &= \mathrm{Tr}(\hat{\rho}) = 1
\end{aligned}$$

Remember that $2\pi\delta(x) = \int_{-\infty}^{\infty} e^{\mathrm{i}\kappa x}\,\mathrm{d}\kappa$ and that $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\mathrm{i}x\,p/\hbar}$.

(iii) f_{W} is Galilei invariant:

$$\begin{aligned}
\psi(q) \to \psi(q+a) &\implies f_{\mathrm{W}}(q,p) \to f_{\mathrm{W}}(q+a,p) \\
\psi(q) \to e^{\mathrm{i}p'q/\hbar}\psi(q) &\implies f_{\mathrm{W}}(q,p) \to f_{\mathrm{W}}(q,p-p')
\end{aligned}$$

Hint: For the density operator $\hat{\rho} = \sum_n p_n|\psi_n\rangle\langle\psi_n|$, we can rewrite the Wigner distribution function as

$$f_{\mathrm{W}}(q,p) = \frac{1}{2\pi\hbar}\sum_n p_n \int_{-\infty}^{\infty}\mathrm{d}y\,\psi_n\left(q-\frac{y}{2}\right)\;\psi_n^*\left(q+\frac{y}{2}\right)e^{\mathrm{i}\frac{py}{\hbar}}.$$

Calculate $\overline{f_{\mathrm{W}}}$ associated to $\overline{\psi_n}(q) = \psi_n(q+a)$ and $\overline{\psi_n}(q) = e^{\mathrm{i}\frac{p'q}{\hbar}}\psi_n(q)$ respectively.

(iv) f_{W} is invariant under space and time reflections:

$$\begin{aligned} \psi(q) \to \psi(-q) \quad &\Longrightarrow \quad f_{\mathrm{W}}(q,p) \to f_{\mathrm{W}}(-q,-p) \\ \psi(q) \to \psi^*(q) \quad &\Longrightarrow \quad f_{\mathrm{W}}(q,p) \to f_{\mathrm{W}}(q,-p) \end{aligned}$$

(v) In the force-free case the equation of motion is the classical one:

$$\frac{\partial f_{\mathrm{W}}}{\partial t} = -\frac{p}{m}\frac{\partial f_{\mathrm{W}}}{\partial q}$$

Consider that the time derivative of the density operator is given by $\dot{\hat{\rho}} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}, \hat{\rho}\right]$ and if no forces are present we can write the Hamiltonian as $\hat{H} = \hat{p}^2/2m$.

3. Weyl representation

Prove that the Weyl representation for $F = q^n p^m$ is

$$\hat{F}_{Weyl} = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\hat{q}^{n-r}\hat{p}^m\hat{q}^r$$

Hint: The associated operator for the classical quantity F is given by Weyl's expansion:

$$\hat{F}_{Weyl}(\hat{q},\hat{p}) = \int \mathrm{d}\sigma \mathrm{d}\tau \, e^{\frac{\mathrm{i}}{\hbar}(\sigma\hat{q}+\tau\hat{p})}\varphi(\sigma,\tau)$$

where

$$\varphi(\sigma,\tau) = \frac{1}{(2\pi\hbar)^2}\int \mathrm{d}q\mathrm{d}p \, e^{\frac{\mathrm{i}}{\hbar}(\sigma q+\tau p)}q^n p^m$$

Frohes Schaffen!