Universität Regensburg Winter Semester 09/10

Mesoscopic Physics

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Sheet 3

1. Wigner distribution for the harmonic oscillator

(a) •Calculate the Wigner distribution

$$f_{\mathrm{W}}(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}y\,\left\langle q-\frac{y}{2}\right|\hat{\rho}\left|q+\frac{y}{2}\right\rangle e^{\mathrm{i}\frac{py}{\hbar}}$$

for a harmonic oscillator. Consider the density operator for pure states and obtain f_{W} for the eigenstates of this system, which we can write as

$$\psi_n(q) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}\left(\frac{1}{2^n n!}\right)^{1/2} e^{\alpha^2 q^2/2} H_n(\alpha q)\,,$$

where H_n is the n-th Hermite polynomial and $\alpha = \sqrt{m\omega/\hbar}$. Use the following equation to derive the final expression for f_{W}

$$\int \mathrm{d}z e^{-z^2} H_n\left(z+\beta+\alpha q\right) H_n\left(z+\beta-\alpha q\right) = 2^n\sqrt{\pi}n! L_n\left(2\left(\alpha^2 q^2-\beta^2\right)\right),$$

L_n being the n-th Laguerre polynomial.

(b) Examining the first Laguerre polynomials, observe the positivity of the ground state and compare it with the excited states. How does it compare to the classical solution?

(c) •A coherent state is constructed from the ground state of the harmonic oscillator by

$$\Psi_z(q) = e^{z\hat{a}^\dagger - z^*\hat{a}}\psi_0(q),$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the ladder operators defined by

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\alpha\hat{q} - \frac{\mathrm{i}\hat{p}}{\hbar\alpha}\right), \quad \hat{a} = \frac{1}{\sqrt{2}}\left(\alpha\hat{q} + \frac{\mathrm{i}\hat{p}}{\hbar\alpha}\right),$$

and z is a complex number. Prove that it can also be written as

$$\Psi_z(q) = e^{\mathrm{i}\frac{p_z q_z}{2\hbar}} e^{-\mathrm{i}\frac{p_z q}{\hbar}}\psi_0(q-q_z),$$

with $p_z = -\sqrt{2}\hbar\alpha\mathrm{Im}(z)$ and $q_z = \sqrt{2}\mathrm{Re}(z)/\alpha$, making use of Glauber's formula $e^{A+B} = e^A e^B e^{-[A,B]/2}$, valid in the case $[[A,B],A] = [[A,B],B] = 0$.

Calculate the Wigner distribution for the coherent states and its time evolution. Sketch the dynamics of the Wigner distribution in phase space. Hint: Start with the time evolution of the generic coherent state and map the result in the Wigner representation.

(d) Write the equation of motion for a Wigner distribution of a harmonic oscillator. Verify that the time dependent Wigner distribution calculated in (c) is a solution.

2. Boltzmann's Magnetotransport

(a) •Solve Boltzmann's equation in the presence of a homogeneous magnetic field $\mathbf{B}$ and a homogeneous electric field $\mathbf{E}$, which in the relaxation time approximation is given by

$$(-e\mathbf{E} - e\mathbf{v}_{\mathbf{k}} \times \mathbf{B}) \frac{\partial f_{\mathbf{k}}}{\partial \hbar \mathbf{k}} = -\frac{f_{\mathbf{k}} - f^0(\varepsilon_{\mathbf{k}})}{\tau_m(\varepsilon_{\mathbf{k}})}$$

Keep only linear terms in $\mathbf{E}$ and make use of the following Ansatz:

$$f_{\mathbf{k}} = f^0(\varepsilon_{\mathbf{k}}) - \frac{\partial f^0(\varepsilon_{\mathbf{k}})}{\partial \varepsilon_{\mathbf{k}}} \Phi_{\mathbf{k}}$$

where $\Phi_{\mathbf{k}} = \mathbf{v}_{\mathbf{k}} \cdot \mathbf{b}$, to transform Boltzmann's equation into the algebraic form

$$e\mathbf{v}_{\mathbf{k}} \cdot \mathbf{E} = -\frac{\mathbf{v}_{\mathbf{k}} \cdot \mathbf{b}}{\tau_m(\varepsilon_{\mathbf{k}})} + \frac{e}{m} \mathbf{v}_{\mathbf{k}} \cdot (\mathbf{B} \times \mathbf{b})$$

for a generic $\mathbf{v}_{\mathbf{k}}$. Derive the solution for $\mathbf{b}$ to obtain

$$\mathbf{b} = -\frac{e\tau_m}{1 + \omega_c^2 \tau_m^2} \left[\mathbf{E} + \frac{\omega_c^2 \tau_m^2}{B^2} (\mathbf{E} \cdot \mathbf{B}) \cdot \mathbf{B} + \frac{\omega_c \tau_m}{B} \mathbf{B} \times \mathbf{E} \right]$$

where $\tau_m = \tau_m(\varepsilon_{\mathbf{k}})$. Remember the triple product expansion: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.

(b) Using once more the previous Ansatz and the result for $\mathbf{b}$, calculate the conductivity σ for the case of $\mathbf{B} = B\hat{\mathbf{z}}$ and an isotropic crystal where $\sigma_{\alpha\beta}(\mathbf{B} = 0) = \delta_{\alpha\beta}\sigma_0$.

Frohes Schaffen!