Universität Regensburg | Wintersemester 10-11

# Density Matrix Theory

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**Sheet 2**

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## 1. Eigenstates, pure states, mixed states

Let us consider a quantum ring described by the Hamiltonian:

$$H = \sum_{\alpha=1}^{N} \left[ \varepsilon c_\alpha^\dagger c_\alpha + b(c_{\alpha+1}^\dagger c_\alpha + c_\alpha^\dagger c_{\alpha+1}) \right]$$

where $c_\alpha^\dagger$ creates a (spinless) electron on the $\alpha$ site and we impose periodic boundary conditions: $c_{N+1} = c_1$.

1. Prove that the single particle eigenvectors $|\ell\rangle$ of the system can be written as:

$$|\ell\rangle \equiv c_\ell^\dagger|0\rangle = \frac{1}{\sqrt{N}} \sum_{\alpha=1}^{N} e^{-i\ell\frac{2\pi}{N}\alpha} c_\alpha^\dagger |0\rangle,$$

   where $\ell = 0 \dots N-1$ and $|0\rangle$ is the vacuum state. Prove also that the corresponding eigenvalues are

$$E_\ell = \varepsilon + 2b\cos\left(\frac{2\pi}{N}\ell\right).$$

2. Calculate the time evolution of the eigenvector $|\ell\rangle$ and prove that after a time interval

$$T = [\varepsilon + 2b\cos(\tfrac{2\pi\ell}{N})]^{-1}\tfrac{2\pi\ell}{N}\hbar$$

   the vector is rotated in space of an angle $2\pi/N$ with respect of the initial vector. Is this rotation physical? What happens if we measure the energy starting from another reference point?
   (Hint: Due to the geometry of the system, a rotation in space of an angle $2\pi/N$ brings the position basis vector $|\alpha\rangle$ into the vector $|\alpha+1\rangle$).

3. Calculate now the time evolution of the pure state $|\ell\rangle\langle\ell|$. Prove that the density matrix is stationary in whatever basis. Comment the result.

4. Consider now the time evolution of the pure state $|\psi\rangle\langle\psi|$, where $|\psi\rangle = a|\ell_1\rangle + b|\ell_2\rangle$ with $\ell_1 \neq \ell_2$ and $|a|^2 + |b|^2 = 1$. Prove that this time the density matrix is evolving in time if $E_{\ell_1} \neq E_{\ell_2}$. Prove that the evolution, at least at finite time intervals can be interpreted as a rotation in space. Find the period of the rotation.

5. Finally consider as an initial condition a mixed state of energy eigenstates: $\rho(t=0) = \sum_{\ell=0}^{N-1} p_\ell |E_\ell\rangle\langle E_\ell|$. Is this density matrix evolving in time?

6. Visualize all the results obtained in the previous points using the time evolution code developed for the previous exercise sheet and extending it to the generic $N$ site system.

**Frohes Schaffen!**