Fakultät für Physik WS 12/13
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Übungen zu Integrierter Kurs II - Festkörper und Statistische Physik

Blatt 7

Übungsleiter:

Dr. Andrea Donarini (3.1.24, phone 2040) (theory, Tue 12h-14h c.t., Phy 7.3.14)
Sebastian Putz (4.1.36, phone 2032) (experiment, Thu 10h-12h c.t., Phy 7.3.14)

Part I: Theory

7.1 Free energy of an ideal Fermi gas

Show that for the ideal Fermi gas the Helmholtz free energy per particle at low temperatures is given by

$$f = \frac{F}{N} = \frac{3\epsilon_F}{5}\left(1 - \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + ...\right),$$

where N is the particle number. (2 points)

7.2 Fermi gas and white dwarfs

White dwarfs are stars in a final stage of their life, and our Sun will become a white dwarf in a couple of billion years.

1. Given that the mass of the sun is $1.99{\cdot}10^{30}$ kg, estimate the number of electrons in the Sun. Assume that the Sun is largely composed of atomic hydrogen. (1 point)

2. In a white dwarf star of one solar mass the atoms are all ionized and contained in a sphere of radius $4{\cdot}10^9$ cm. Find the Fermi energy of both the electrons and the nucleons in eV units. Is a non-relativistic treatment of the electrons and nucleons, respectively, justified? (2 points)

3. If the temperature of the white dwarf is 10^7 K, discuss whether the electrons and/or the nucleons in the star are degenerate. (2 points)

7.3 Stability of a white dwarf against gravitational collapse

It is energetically favorable for a body held together by gravitational force to be as compact as possible. We take a star to be made up of an approximately equal number N of electrons and protons, since otherwise the Coulomb interaction would overcome the gravitational interaction. Somewhat arbitrarily we also assume that these is an equal number of neutrons and protons. On Earth the gravitational pressure is not large enough to overcome the repulsive forces between atoms and molecules at short distance. Inside the Sun the matter is not in the form of atoms and molecules, but since it is still burning, the radiation pressure keeps it from collapsing. Let us consider a burnt out star such as a white dwarf. Assume that the temperature of the star is low enough compared to the Fermi temperature, that the electrons can be considered as a $T = 0$ Fermi gas. Because of their large mass the kinetic energy of protons and neutrons is small compared to that of the electrons.

1. Show that if the electron gas is non-relativistic, the electron mass is m_e and the radius of the star is R, the electron kinetic energy of the star can be written (1 point)

$$E_{\text{kin}} = \frac{3\hbar^2}{10m_e}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}}{R^2}.$$

2. The gravitational potential energy is dominated by the neutrons and protons. Let m_N be the nucleon mass. Assume the mass density is approximately constant inside the star. Show that if there is an equal number of protons and neutrons, the potential energy is given by

$$E_{\text{pot}} = -\frac{12}{5}m_N^2 G\frac{N^2}{R},$$

where $G = 6.67 \cdot 10^{-11}$ Nm^2/kg^2 is the gravitational constant. (2 points)

3. Find the radius for which the potential energy plus kinetic energy is a minimum for a white dwarf with the same mass as the Sun ($m_\odot = 1.99 \cdot 10^{30}$ kg), in units of the radius of the Sun ($R_\odot = 6.96 \cdot 10^8$ m). (2 points)

7.4 Rearrangement Theorem

The rearrangement theorem states that for the set of elements $\{g_i\}$ forming a group, if each element is multiplied from the left, or from the right, by a particular element g_j of $\{g_i\}$, then the set $\{g_i\}$ is regenerated with the elements, in general, re-ordered. Prove this theorem making use of the properties of a group, showing first that every element of the group is contained and then that it is contained only once. (1 point)

7.5 Groups and subgroups

1. Show that, with multiplication as binary composition, the set $\{1\ -1\ i\ -i\}$ where $i^2 = -1$, form a group G.

2. A subset $H \subset G$, that is itself a group with the same law of binary composition, is a subgroup of G. That is, H has to satisfied closure as all other properties are automatically fulfilled. Find the subgroups of G.

7.6 Symmetry operations

Consider the molecule AB_4, where the B atoms lie at the corners of a square and the A atom is at the center and is not coplanar with the B atoms.

1. Determine the symmetry operations for this molecule.

2. Find its multiplication table.

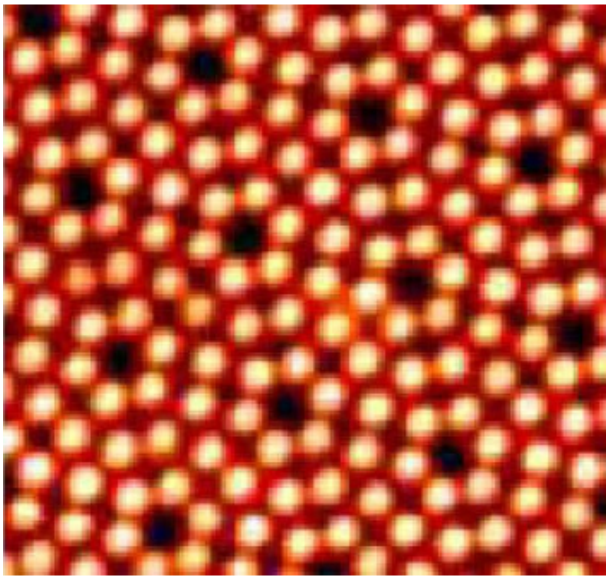
3. List the subgroups.

4. List the classes.

Part II: Experiment

7.7 Symmetrieelemente und Millersche Indizes

Die Abbildung zeigt eine Oberfläche eines Silizium-Einkristalls, abgebildet mit einem Rastertunnelmikroskop. Überlegen Sie sich Symmetrieoperationen, die die abgebildete Oberfläche in sich selbst überführen würden, wie z.B. Spiegelebenen oder Drehachsen. Um welche Kristallebene handelt es sich bei der abgebildeten Struktur (Angabe der Millerschen Indizes)? (2 Punkte)



7.8 Gitterbedingung für Bravais-Gitter

Drei nicht-komplanare Vektoren $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ spannen im dreidimensionalen Raum ein Parallelepiped (eine *primitive Elementarzelle*) auf. Durch fortlaufende Aneinanderfügung dieser Translationsvektoren bzw. Parallelepipede entsteht ein Gitter, dessen Gitterpunkte die Eckpunkte der Parallelepipede sind. Ein solches Gitter nennt man Translations- oder Bravais-Gitter. Für jedes dieser Gitter gilt die folgende Gitterbedingung: Alle Gitterpunkte haben eine identische (also insbesondere auch gleich orientierte) Umgebung. Zeigen Sie, dass diese Gitterbedingung beim kubisch flächenzentrierten, wie auch beim kubisch raumzentrierten Gitter erfüllt ist, nicht aber bei einem kubischen Gitter mit gleichzeitiger Flächen- und Raumzentrierung. Dieses Letztere ist also kein Bravais-Gitter. (2 Punkte)

7.9 Kubische Bravais-Gitter

Bestimmen Sie für die drei kubischen Bravais-Gitter (primitiv, raumzentriert, flächenzentriert) mit Gitterkonstante a folgende charakteristische Grssen:

1. Zahl der Gitterpunkte pro kubischer Elementarzelle (1 Punkt)
2. Abstand der nächsten Nachbarn (1 Punkt)
3. Zahl der nächsten Nachbarn (*Koordinationszahl*) (1 Punkt)
4. Abstand und Zahl der zweitnächsten Nachbarn (1 Punkt)
5. Volumenverhältnis der kubischen zur primitiven Elementarzelle (1 Punkt)