Universität Regensburg Winter Semester 14/15

Quantum Theory of Condensed Matter II

Mesoscopic Physics

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Fri at 12:30, in 5.0.20

Sheet 1

1. Brownian motion in one dimension

The Brownian motion concerns a heavier particle immersed into a bath of lighter ones (*e.g.* pollen grains in water, like in the experiment performed by Brown in 1827). The mutual collisions between the different particles can be detected by observing the motion of the heavy one: small random leaps combining into a diffusive motion. Due to the randomness of the actual trajectory, a statistical treatment of the problem is the most appropriate.

Let us treat for simplicity the one dimensional case. Consider the motion of a particle on a line done in discrete (*i.e.* always of size a) random leaps starting from the origin. In particular a leap to the left happens with probability p and one to the right with probability q such that $1 = q + p$.

1. Assume that each leap is done independently of the earlier history of the motion. Prove that the probability to have done n_R leaps to the right out of a total of N leaps reads:

$$W_N(n_R) = \frac{N!}{n_R!(N-n_R)!} p^{N-n_R} q^{n_R}.$$

Hint: In order to obtain the prefactor consider all the different combination of the leaps to the left and to the right. Check that $W_N(n_R)$ is a well defined probability distribution by proving the normalization condition

$$\sum_{n_R=0}^{N} W_N(n_R) = 1$$

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(2 Points)

2. The position of the heavy particle is given by $m = (n_R - n_L)a$ where n_L is the number of leaps to the left. Using the probability distribution introduced at previous point, calculate the mean position of the heavy particle after N leaps. What is the result for the case $p = q = 1/2$? Give a physical interpretation of the result.

Hint: Remember that the leaps can only happen either to the left or to the right thus $n_L = N - n_R$. Moreover, recall the mean value formula:

$$\langle O \rangle_N = \sum_{n_R=0}^{N} O(n_R) W_N(n_R).$$

The following formula could also be of interest:

$$(p+q)^N = \sum_{n=0}^{N} \frac{N!}{n!(N-n!)} q^n p^{N-n}.$$

(2 Points)

3. Calculate the mean square distance covered by the heavy particle in N leaps. Prove that in the case $p = q = 1/2$ one obtains:

$$\langle \Delta m^2 \rangle = \langle (m - \langle m \rangle)^2 \rangle = N a^2.$$

(2 Points)

2. Diffusion equation

Here, we want to address the continuous limit of the distribution $W_N(n)$ introduced in the previous exercise.

1. Consider the distribution $W_N(n)$ with $q = p = 1/2$. Argue that, as a function of n, $W_N(n)$ has a (sharp) maximum around $n \approx N/2$. Consequently prove that, in the limit $N \gg 1$, one can write

$$W_N(n) \approx \frac{1}{\mathcal{N}} \exp\left[-\frac{1}{2N}(2n - N)^2\right],$$

where $\mathcal{N}$ is a normalization factor independent of n.

Hint: Consider the logarithm of the distribution and Taylor expand it around its maximum. The following relation could be of interest: $\ln M! \approx M \ln M - M$ (Stirling's approximation).

(2 Points)

2. Let us now assume that a random leap happens every time interval δt and perform the simultaneous limit $\delta t \to 0$ and $N \to \infty$ by keeping constant the product $N\delta t = t$. Prove that $W_N(n) \to w(x, t)$, where

$$w(x, t) = \frac{1}{\sqrt{4\pi t D}} \exp\left(-\frac{x^2}{4tD}\right),$$

with $x = (2n - N)a$ being the position of the heavy particle, and $D = \frac{a^2}{2\delta t}$. Notice that in order to keep D finite also the size a of the leap must be infinitesimal. Nevertheless $Na \to \infty$. Can you prove it? As a consequence $w(x, t)$ is defined on the entire (one dimensional) space.

(2 Points)

3. Using the distribution derived at the previous point prove the following relations:

$$\begin{aligned} \langle x(t) \rangle &= 0, \\ \langle x^2(t) \rangle &= 2Dt. \end{aligned}$$

(1 Point)

4. Verify that the distribution $w(x, t)$ solves the diffusion equation as obtained from the continuity relation and Fick's law:

$$\frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2},$$

thus identifying the meaning of the constant D.

(1 Point)

3. Diffusion and velocity correlators

In this exercise we will work out the relation between the diffusive dynamics discussed in the previous exercises and the velocity correlator.

1. Consider an ensemble of Brownian particles moving in one dimension and prepared in the origin at time $t = 0$. Prove that the following relation holds:

$$\langle x(t)^2 \rangle = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \langle v(t')v(t'') \rangle,$$

where $v(t)$ is the velocity of one arbitrary particle at time t and $\langle v(t')v(t'') \rangle$ is the velocity correlator.

(2 Points)

2. Due to the observation that the heavy particle is in equilibrium with the bath of lighter ones we can argue that $\langle v(t')v(t'') \rangle = \langle v(t' - t'')v(0) \rangle$. Using this relation prove that

$$\langle x(t)^2 \rangle = 2 \int_0^t \mathrm{d}s (t - s) \langle v(s)v(0) \rangle.$$

(2 Points)

3. Assuming that $C_v(s) = \langle v(s)v(0) \rangle$ is an exponentially decaying function of s, $C_v(s) = v_0^2 e^{-s/\tau}$, prove that, for $t \gg \tau$, it holds

$$\langle x(t)^2 \rangle = 2Dt.$$

where $D = v_0^2 \tau$. Compare the result with the one obtained in exercise 2. Argue why the specific form of C_v does not influence qualitatively the result as far as its decay in time is sufficiently fast.

(2 Points)

Frohes Schaffen!